

IASSNS-HEP-95/25
 UFIFT-HEP-95-26
 NSF-ITP-95-130
 hep-th/9510223
 October 1995

String Unification, Higher-Level Gauge Symmetries, and Exotic Hypercharge Normalizations

Keith R. Dienes^{1,3*}, Alon E. Faraggi^{1,2,3†}, and John March-Russell^{1‡}

¹ *School of Natural Sciences, Institute for Advanced Study
Olden Lane, Princeton, NJ 08540 USA*

² *Department of Physics, University of Florida
Gainesville, FL 32611 USA[§]*

³ *Institute for Theoretical Physics, University of California
Santa Barbara, CA 93106 USA[¶]*

Abstract

We explore the extent to which string theories with higher-level gauge symmetries and non-standard hypercharge normalizations can reconcile the discrepancy between the string unification scale and the GUT scale extrapolated from the Minimal Supersymmetric Standard Model (MSSM). We determine the phenomenologically allowed regions of (k_Y, k_2, k_3) parameter space, and investigate the proposal that there might exist string models with exotic hypercharge normalizations k_Y which are less than their usual value $k_Y = 5/3$. For a broad class of heterotic string models (encompassing most realistic string models which have been constructed), we prove that $k_Y \geq 5/3$. Beyond this class, however, we show that there exist consistent MSSM embeddings which lead to $k_Y < 5/3$. We also consider the constraints imposed on k_Y by demanding charge integrality of all unconfined string states, and show that only a limited set of hypercolor confining groups and corresponding values of k_Y are possible.

* E-mail address: dienes@sns.ias.edu

† E-mail address: faraggi@phys.ufl.edu

‡ E-mail address: jmr@sns.ias.edu

§Current address.

¶Temporary address.

1 Introduction and Summary

String theory [1] is a unique candidate for the consistent unification of quantum gravity with the gauge interactions. Moreover, as a fundamental theory of the elementary interactions, string theory predicts a natural unification of the corresponding couplings [2, 3]. The scale at which such a unification takes place is related to the Planck scale, and at the one-loop level is found to be of the order [4]:

$$M_{\text{string}} \approx g_{\text{string}} \times 5 \times 10^{17} \text{ GeV}. \quad (1.1)$$

However, if one assumes that the particle spectrum between the electroweak scale and the unification scale is that of the Minimal Supersymmetric Standard Model (MSSM), then the low-energy data predict a unification of the gauge couplings at a scale which is of the order

$$M_{\text{MSSM}} \approx 2 \times 10^{16} \text{ GeV}. \quad (1.2)$$

Thus, an order of magnitude separates the string unification scale and the MSSM unification scale. In other words, string-scale unification — together with the hypothesis that the spectrum below the string scale is that of the MSSM — predicts values for $\sin^2 \theta_W(M_Z)$ and $\alpha_{\text{strong}}(M_Z)$ which strongly disagree with the experimentally observed values. This is the well-known problem of string-scale gauge-coupling unification, which is one of the more important issues facing string phenomenology.

It would seem that in an extrapolation of the gauge parameters over fifteen orders of magnitude, a problem involving a single order of magnitude would have many possible resolutions. Indeed, in string theory there are many possible effects that can *a priori* account for the discrepancy, and a general discussion of these effects can be found in Refs. [5, 6]. For example, there could be large one-loop threshold corrections due to the infinite towers of heavy string states which are otherwise neglected in an analysis of the purely low-energy (massless) string spectrum. Such threshold corrections could redefine the string unification scale, and potentially produce a new effective scale in agreement with the MSSM unification scale. Alternatively, one could contemplate that additional matter or gauge structure beyond the MSSM could reconcile the two scales. Another possibility, even within the MSSM gauge group and matter structure, is that the boundary conditions for the gauge couplings at unification could be altered by realizing the $SU(2)$ and $SU(3)$ gauge factors of the MSSM as affine Lie algebras with levels $k_2, k_3 > 1$. A final possibility is that the normalization k_Y of the weak hypercharge current $U(1)_Y$ in string theory is not necessarily the one that is traditionally found in grand unified theories (GUT's). Hence, the potential exists for non-standard hypercharge normalizations to alter the boundary conditions of the gauge couplings at unification in such a way as to resolve the discrepancy between the two unification scales.

Despite these many possible approaches, however, it turns out that the gauge coupling unification problem is not simple to resolve within realistic string models.

While an analysis of the moduli dependence of the string threshold corrections [7] shows that these corrections can indeed be substantial for large moduli, a general expectation is that the string moduli settle near the self-dual point, and are therefore naturally small [8]. For small moduli, many numerical investigations [4, 9, 6] have shown that the string threshold corrections are small, and thus do not significantly affect the string unification scale. Indeed, in Ref. [6], general arguments were given that explain the suppression of the string threshold correction in any modular-invariant string theory without large moduli. Thus, without large moduli (or large products of moduli*), string threshold corrections are not likely to resolve the problem.

An alternative scenario is to contemplate the existence of a simple GUT symmetry group between the MSSM unification scale and the string scale. However, despite several efforts [12], no viable model employing this option has been constructed to date. Another possibility arises if there exists additional gauge structure beyond the MSSM, arising at an intermediate energy scale. Such additional gauge structure appears, for example, in string models [13] realizing the flipped $SU(5)$ scenario [14], or in string models [15] realizing the Pati-Salam scenario [16]. Such additional gauge structure may also appear in the form of extended custodial symmetries that are peculiar to certain string models [17]. Surprisingly, however, in all of these examples, a careful analysis shows that the additional gauge structure at intermediate energy scales only *increases* the disagreement with the experimental data [5, 6].

Yet another possibility is that there exists additional non-MSSM matter in the desert between the electroweak scale and the string unification scale [18, 19, 20]. In fact, the availability of such additional thresholds in realistic free-fermionic models was demonstrated in Ref. [18]. Thus, in this respect, imposing the restriction that the spectrum below the string scale is just that of the MSSM is *ad hoc*, and may be too restrictive. A detailed analysis of these matter states then shows that while some of the realistic string models can achieve successful string-scale unification if their non-MSSM matter states arise at appropriate intermediate mass scales, in other models such matter cannot resolve the discrepancy [5, 6].

A final suggestion [21] is that in string theory, the normalization of the weak hypercharge can generally differ from that in grand-unified theories. Indeed, if this hypercharge normalization k_Y is carefully chosen relative to the non-abelian MSSM levels (k_2, k_3), the three gauge couplings will approximately unify at the string scale rather than at the MSSM scale. This is clearly a very elegant and economical resolution of the problem. Indeed, this scenario is as predictive as usual GUT unification, since the introduction of the new string parameter k_Y is offset by the new string-predicted relation (1.1) between the gauge coupling at unification and the unification scale. However, it is found that in order for this scenario to work within level-one

* For $(2, 0)$ string compactifications, it is effectively a *product* of moduli that determines the size of threshold corrections [10]. It is then found [11] that suitably chosen small values for individual moduli have the potential to yield larger threshold corrections. Unfortunately, this mechanism has not yet been realized within the context of realistic string models.

string models, this weak hypercharge normalization k_Y should have the approximate value $k_Y \approx 1.4$. Unfortunately, despite various attempts [22], no models with k_Y in this range have yet been constructed. Furthermore, as we shall see, models with k_Y in this range would have immediate problems with fractionally charged states, and it is therefore unclear whether such models would be realistic.

It is an important observation that hypercharge normalizations $k_Y \approx 1.4$ are smaller than the normalization factor $k_Y = 5/3$ which appears in simple grand-unified theories such as $SU(5)$ or $SO(10)$. An important question, therefore, is whether values $k_Y < 5/3$ are realizable in string models, or whether there is some fundamental reason why they are not. Of course, from the point of view of the underlying (rational) worldsheet conformal field theory, k_Y is indeed a free (rational) parameter, and there are no further theoretical restrictions on the allowed values of k_Y . However, within a consistent string theory, other constraints can appear.

One such constraint arises due to the hypercharge of the positron. In the Standard Model, the positron is a color- and electroweak-singlet. Thus, the sole contribution to the conformal dimension of the positron state from the Standard Model gauge group factors is solely that from its weak hypercharge. From this fact and the fact that the positron must be realized as a massless string state, it is possible to show [23] that any phenomenologically consistent string model must have $k_Y \geq 1$.

The question then arises as to whether other phenomenological constraints can also be imposed on the value of k_Y . A general expectation might be that $k_Y = 5/3$ is the minimal value allowed, simply because the standard $SO(10)$ embedding is an extremely economical way to embed all three generations with universal weak hypercharge assignment. Indeed, all of the realistic string models constructed to date have $k_Y \geq 5/3$. However, the possibility remains that there might exist special, isolated string models or constructions which manage to circumvent this bound, and populate the range $1 \leq k_Y < 5/3$.

In this paper, we shall undertake a general investigation of these issues. Rather than look at specific string constructions, we shall generally explore the extent to which string theories can accommodate the higher-level gauge symmetries and/or non-standard hypercharge normalizations that would be necessary in order to reconcile the discrepancy between string-scale unification and low-energy couplings. Thus, we shall attempt a systematic analysis of the constraints that govern which values of (k_Y, k_2, k_3) are mutually realizable in consistent realistic string models.

We begin by carefully analyzing the renormalization group equations in order to determine the phenomenologically preferred regions of (k_Y, k_2, k_3) parameter space. We find, as expected, that values $k_Y/k_2 < 5/3$ are preferred on the basis of gauge coupling unification, and that given the experimentally observed low-energy couplings, only a narrow band of values for $(k_Y/k_2, k_3/k_2)$ are able to achieve string-scale unification without large corrections from other sources. However, we also find a surprisingly strong correlation between these *ratios* of the levels (k_Y, k_2, k_3) , and their absolute sizes. For example, we find that choosing the ratio $k_Y/k_2 \approx 1.45 - 1.5$

is roughly consistent with relatively small absolute sizes for the levels (k_2, k_3) , but choosing $k_Y/k_2 \approx 1.4$ requires levels (k_2, k_3) which are significantly greater. These results should therefore provide strong constraints generally applicable to realistic string model-building employing higher-level gauge symmetries and/or non-standard hypercharge normalizations.

Given these general results, we then focus on the values that k_Y/k_2 may take in realistic string models. We follow essentially two “orthogonal” lines of approach.

Our first approach towards studying the possible values of k_Y that may arise for a given hypercharge group factor involves analyzing the allowed embeddings of that hypercharge into a consistent charge lattice. Specifically, by imposing the phenomenological requirement that the entire MSSM spectrum appear and have the correct hypercharge assignments, we can examine the possible embeddings of the weak hypercharge in terms of fundamental worldsheet currents. In this way we are able to provide various constraints on the value of k_Y . We prove, for example, that for a large class of realistic level-one string models employing the so-called “minimal” MSSM embeddings, one must have $k_Y \geq 5/3$. This class includes most realistic free-field string models which have been constructed in the literature. However, it is nevertheless possible to extend such an analysis beyond embeddings in this class, and we find that we can construct consistent hypercharge embeddings which manage to have $k_Y/k_2 < 5/3$ by using higher-twist sectors and/or higher levels (k_2, k_3) for the non-abelian $SU(2) \times SU(3)$ MSSM gauge factors. The special hypercharge embeddings that we construct can therefore serve as a useful starting point in the construction of realistic $k_Y/k_2 < 5/3$ string models.

Our second approach is complementary to the first, and instead examines the role that modular invariance plays in restricting the mutually realizable values of (k_Y, k_2, k_3) . As is well-known, string models typically contain a plethora of unwanted massless chiral states carrying fractional electric charges, and it is important for the purposes of realistic string model-building to be able either to avoid such states altogether, or to arrange to have them suitably confined under the influence of an additional “hypercolor” gauge interaction. However, it turns out [23] that the possible scenarios by which this can be done are, through modular invariance, highly dependent on the values of (k_Y, k_2, k_3) . It is therefore important to *correlate* the allowed values of (k_Y, k_2, k_3) with the appearance of fractionally charged states in the string spectrum, and with the extra gauge interactions under which such states might confine. For example, it is straightforward to show that any level-one $SU(3) \times SU(2) \times U(1)$ string model *without* fractionally charged states must have $k_Y > 5/3$. In this paper we shall analyze the more general situation in which fractionally charged states can appear and are confined. In this way we shall determine for which classes of fractionally charged states and confining groups the phenomenologically preferred values of $k_Y/k_2 < 5/3$ can be realized.

This paper is organized as follows. In Sect. 2, we provide a self-contained review of the definition of k_Y in string theory, and the methods by which it is calculated.

In Sect. 3, we then analyze the renormalization group equations in order to determine the phenomenologically preferred regions in (k_Y, k_2, k_3) parameter space. We then proceed, in Sects. 4 and 5, to investigate the possible hypercharge embeddings that can be realized in string theory. Sect. 4, in particular, contains our proof that $k_Y \geq 5/3$ within a broad class of realistic string models employing the “minimal” hypercharge embeddings, and in Sect. 5 we go beyond this class in order to construct consistent hypercharge embeddings with $k_Y/k_2 < 5/3$. We then turn, in Sect. 6, to the constraints imposed on (k_Y, k_2, k_3) by demanding the charge integrality of unconfined string states, and in Sect. 7 we conclude with a summary of our results and comments regarding various extensions. Two background discussions have also been collected in Appendices. Appendix A contains a review of the standard hypercharge assignments for the MSSM states, and Appendix B contains the derivation of a result quoted in Sect. 3 concerning the level-dependence of two-loop corrections to the renormalization group equations.

2 Technical Background: Definition of k_Y

Despite its seemingly simple role as the hypercharge normalization, the factor k_Y in string theory turns out to have an unexpectedly subtle definition and interpretation. In this section, therefore, we shall give a technical review of this issue, and establish the normalizations that we shall use throughout this paper.

The normalization factor k_Y for the hypercharge group factor $U(1)_Y$ is similar in many ways to the so-called “levels” that appear in the more general case of non-abelian untwisted affine Lie algebras. We begin, therefore, by reviewing some basic facts concerning these algebras, also known as Kač-Moody algebras [24]. Kač-Moody algebras \hat{G} are infinite-dimensional extensions of the ordinary Lie algebras G , and contain the ordinary Lie algebras as subalgebras. They are generated by chiral world-sheet currents $J^a(z)$ of conformal dimension $(1, 0)$ satisfying the operator product expansions (OPE’s)

$$J^a(z)J^b(w) = \frac{\tilde{k} \delta_{ab}}{(z-w)^2} + \frac{i f^{abc}}{z-w} J^c(w) + \text{regular} \quad (2.1)$$

where f^{abc} are the structure constants of the Lie algebra G [with $a, b, c = 1, \dots, \dim(G)$], and where the first term on the right side (the “central extension” or Schwinger term with coefficient \tilde{k}) is the new feature characterizing the Kač-Moody algebra. It is clear that the special case with $\tilde{k} = 0$ corresponds to the ordinary Lie current algebra.

For a non-abelian group (*i.e.*, one with non-vanishing structure constants f^{abc}), one can define a unique normalization for the currents $J^a(z)$ by fixing a particular normalization for the structure constants and then demanding that the currents satisfy relations of the form (2.1). One typically specifies the normalization of the

structure constants via

$$\sum_{ab} f^{abc} f^{abd} = C_G^{(\text{adj})} \delta^{cd} \quad (2.2)$$

where $C_G^{(\text{adj})}$ is the eigenvalue of the quadratic Casimir acting on the adjoint representation. This then fixes the normalizations of the currents, and by extension fixes the value of the central extension coefficient \tilde{k} as well as the lengths of the root vectors $\{\vec{\alpha}\}$ of the corresponding Lie algebra. Alternatively, one can define the normalization-independent quantities

$$\tilde{h}_G \equiv \frac{C_G^{(\text{adj})}}{\vec{\alpha}_h^2}, \quad k_G \equiv \frac{2\tilde{k}}{\vec{\alpha}_h^2} \quad (2.3)$$

where $\vec{\alpha}_h$ is the longest root. Here \tilde{h}_G is the “dual Coxeter number” of the group G , and k_G is the “level” of the Kač-Moody algebra. We see, then, that the level k_G of a Kač-Moody algebra has invariant (intrinsic) meaning only for a *non-abelian* group G . The central charge of the corresponding conformal theory can then be similarly expressed:

$$c_G = \frac{k \dim(G)}{k + \tilde{h}_G}, \quad (2.4)$$

and likewise the conformal dimension of any given representation of that algebra is given by:

$$h_{(R)} = \frac{C_G^{(R)} / \vec{\alpha}_h^2}{k + \tilde{h}_G} \quad (2.5)$$

where $C_G^{(R)}$ is the eigenvalue of the quadratic Casimir acting on the representation R . This eigenvalue is defined analogously to (2.2):

$$\sum_{a=1}^{\dim(G)} (T^a T^a)_{ij} = C_G^{(R)} \delta^{ij} \quad (2.6)$$

where T^a are the group generators corresponding to the representation R .

Given these results for non-abelian groups, we now turn to the “Kač-Moody levels” k_Y associated with abelian $U(1)$ groups. In the case of the weak hypercharge $U(1)_Y$, the corresponding value of k_Y plays a crucial role in string theory because, like the values of k_3 and k_2 corresponding to the color and electroweak groups $SU(3)$ and $SU(2)$, this value affects the boundary conditions of the gauge couplings at the string scale, and consequently affects the predicted values of low-energy parameters such as $\sin^2 \theta_W$ and α_{strong} . As we shall see, however, the situation is somewhat more complicated for such abelian group factors.

Strictly speaking, it is impossible to define the intrinsic normalization-independent “level” k at which an abelian group is realized. The difficulty arises as follows. As we have discussed, the Kač-Moody level of a non-abelian group is uniquely defined

as $k \equiv 2\tilde{k}/\vec{\alpha}_h^2$ where $\vec{\alpha}_h$ is the longest root, and where \tilde{k} is the coefficient appearing in the OPE's of its currents, as in (2.1). In the case of an *abelian* factor, however, the structure constants f^{abc} all vanish, and consequently we would find the corresponding OPE

$$J(z)J(w) \stackrel{?}{=} \frac{k \vec{\alpha}_h^2/2}{(z-w)^2} + \text{regular} . \quad (2.7)$$

However, an abelian group factor contains no non-zero roots, and therefore there is no longest root $\vec{\alpha}_h$ which can be used to set a scale for this double-pole term or to absorb changes in the normalization of the $U(1)$ current J . Stated equivalently, we are prevented from fixing a normalization for the currents by the absence of structure constants in the algebra. We see, therefore, that there is no invariant way of defining the intrinsic Kač-Moody level of an abelian gauge group factor. Consequently, unlike the levels of non-abelian group factors, the definition of k_Y (and with it the normalization of the corresponding $U(1)$ current J) becomes a matter of *convention*.

The convention which is typically chosen makes use of the fact that we are realizing such Kač-Moody algebras in the particular physical context of string theory. For *non-abelian* currents J^a , the normalizations are conventionally fixed through the gauge couplings of these currents to each other (through the three and four gauge boson vertices), and through their couplings to gravity (through the gauge-gauge-graviton vertex). The corresponding Kač-Moody level k then turns out to be the ratio between these couplings [2]. Indeed, this relative factor of k_i arises because the three gauge boson contributions arise from the single-pole terms in the OPE (2.1), while the gauge-graviton contributions arise from the double pole. We thereby obtain a relation of the form [2]

$$8\pi \frac{G_N}{\alpha'} = k_i g_i^2 \quad (2.8)$$

between the gravitational (Newton) constant G_N and the gauge coupling g_i of any non-abelian group factor.

For an *abelian* current, however, we have no three gauge boson couplings. We nevertheless continue to have couplings to gravity. We therefore fix a proper normalization for a given $U(1)$ current J by demanding that it have the same coupling to gravity as any non-abelian current. As shown in Ref. [2], this is tantamount to requiring that J take a normalization giving rise to the OPE

$$J(z)J(w) = \frac{1}{(z-w)^2} + \text{regular} . \quad (2.9)$$

An important point to notice here is that this is a *fixed* normalization which is independent of the lengths of the roots of any of the non-abelian group factors. Of course, we expect on physical grounds that this must be the case, for it is possible to build string models for which there are no non-abelian gauge group factors. Such models will nevertheless exhibit couplings to gravity, however, and therefore such a universal normalization for the $U(1)$ currents can always be defined.

Given the normalization (2.9) for a proper $U(1)$ current, the next issue is to define the “level” k_1 to which it corresponds. As we have already discussed, there is no invariant way of defining the level from the algebra itself [*e.g.*, we cannot use a relation of the form (2.7)], and therefore we instead define the “level” k_1 based on physical terms in analogy with the relation (2.8). In particular, given a $U(1)$ gauge group factor whose current is normalized according to (2.9), we can calculate the ratio of the corresponding gauge coupling g_1 to the gravitational coupling. Such a calculation yields

$$8\pi \frac{G_N}{\alpha'} = 2 g_1^2 . \quad (2.10)$$

Therefore, by comparison with (2.8), we define $k_J \equiv 2$ for any current normalized according to (2.9). Note that this then fixes a scale for the overall definition of the “level” k_1 corresponding to *any* $U(1)$ current with *any* arbitrary normalization. In particular, the factor k_J corresponding to such an arbitrarily normalized current J can be easily determined from the leading coefficient in its OPE expansion:

$$J(z)J(w) = \frac{k_1/2}{(z-w)^2} + \text{regular} . \quad (2.11)$$

We may therefore regard (2.11) as the general definition of the “level” k_1 of any $U(1)$ current J .

Given this definition, it is then straightforward to determine the levels k_1 and k_G of each abelian or non-abelian gauge group factor that arises in a given string model. Recall that in a given heterotic string model, there are typically 22 worldsheet currents which comprise the the Cartan subalgebra of the total gauge group. In a free-fermionic model-construction procedure, for example, these elementary worldsheet Cartan currents \hat{J}_i are in one-to-one correspondence with the internal worldsheet complex fermions f_i , with $\hat{J} \equiv f_i^* f_i$ ($i = 1, \dots, 22$). Likewise, in a bosonic construction, these Cartan currents are in one-to-one correspondence with the internal worldsheet bosons ϕ_i , with $\hat{J}_i \equiv i\partial\phi_i$. Now, in some string models, these Cartan currents $i\partial\phi_i$ may combine with other non-Cartan currents $\exp(i\alpha_i\phi_i)$ to fill out the adjoint representation of a non-abelian gauge group; in such cases the corresponding level k_G can be directly computed via (2.1) and (2.3). By contrast, there may also be various Cartan currents which do *not* combine with any non-Cartan currents; these then give rise to elementary $U(1)$ factors in the gauge group. However, since these Cartan currents \hat{J}_i are always normalized to satisfy (2.9), we see that any such elementary $U(1)$ gauge factors will have $k_1 = 2$.

Similar considerations also hold for $U(1)$ gauge group factors whose currents are realized as *linear combinations* of the elementary Cartan generators. Indeed, in realistic string models, the gauge group factor $U(1)_Y$ corresponding to the Standard Model hypercharge will typically arise as such a linear combination. The corresponding values of k_Y can nevertheless be determined in the manner discussed above. In particular, let us suppose that the hypercharge current Y corresponding to the $U(1)_Y$

gauge factor is comprised of the elementary worldsheet Cartan currents \hat{J}_i via a linear combination of the form

$$Y = \sum_i a_i \hat{J}_i \quad (2.12)$$

where the a_i are certain model-specific coefficients which describe the *embedding* of the physical weak hypercharge current in terms of the Cartan currents. Of course, these coefficients a_i and their overall normalization must be chosen so that the eigenvalues of Y will agree with the usual hypercharge assignments for the MSSM states in the string spectrum (see Appendix A). However, since the elementary currents \hat{J}_i are each individually normalized so as to satisfy (2.9), we see from (2.12) that the hypercharge current Y will have an OPE of the form

$$Y(z)Y(w) = \frac{\sum_i a_i^2}{(z-w)^2} + \text{regular} . \quad (2.13)$$

Comparing with (2.11) then allows us to identify the corresponding “level”:

$$k_Y \equiv 2 \sum_i a_i^2 . \quad (2.14)$$

Thus, in this way, we can define the level k_Y corresponding to the hypercharge $U(1)_Y$ gauge group factor in a given string model. Of course, as explained above, if $k_Y \neq 2$ we must subsequently renormalize Y so that the properly normalized hypercharge current

$$\hat{Y} \equiv \sqrt{\frac{2}{k_Y}} Y \quad (2.15)$$

satisfies the conventional OPE (2.9) with $k_{\hat{Y}} = 2$. In any case, it is clear from (2.14) that

$$k_{cY} = c^2 k_Y \quad (2.16)$$

for any rescaling factor c , and thus such rescalings are straightforward.

It is also straightforward to determine the conformal dimension of a state with a given hypercharge. For such an abelian group factor $U(1)_Y$, the conformal dimension formula (2.5) directly reduces to

$$h = \frac{Y^2}{k_Y} . \quad (2.17)$$

However, it is important to verify by some independent means that the result (2.5) holds even for an abelian group. This is most easily done by starting from the usual result $h = Q^2/2$ for properly normalized $U(1)$ charges such as \hat{Y} , and then rescaling \hat{Y} as in (2.15) to yield the result (2.17) for Y . As required, this result for the conformal dimension is invariant under rescalings of Y as a consequence of (2.16).

Before concluding, we emphasize once again that this procedure for identifying the “level” k_Y of the $U(1)_Y$ hypercharge gauge group factor is ultimately a matter of

convention, for there is no intrinsic method of defining the Kač-Moody level in the case of an abelian group factor. Consequently, unlike the case of non-abelian group factors, *we are not able to uniquely determine the value of k_Y without reference to the string model in which this group factor is ultimately realized*. Indeed, k_Y is not an intrinsic function of the spacetime gauge symmetries and associated algebras, but instead depends on the *embedding* of these symmetries within a consistent string model. Thus, the determination of k_Y is highly model-dependent, and in principle any rational value of k_Y may be obtained. There are, however, certain immediate bounds that can be placed on the value of k_Y . For example, given the MSSM hypercharge assignments listed in Appendix A, we see that applying (2.17) to the $Y = 1$ positron singlet state and requiring that this state be massless (so that $h \leq 1$) trivially shows that we must have $k_Y \geq 1$ in any string model containing such a massless state. One of the main goals of this paper is to analyze what other constraints may be placed on the possible values of k_Y for realistic models containing the MSSM gauge group and spectrum.

3 String Unification and Higher-Level Gauge Symmetries: A Renormalization-Group Analysis

We now explore the extent to which string-scale unification might be reconciled with low-energy data by choosing appropriate values for the three Kač-Moody levels (k_Y, k_2, k_3) which govern the MSSM gauge group. As we shall see, there are a variety of constraints that come into play, and only a limited number of possibilities are phenomenologically viable.

As we have discussed in Sect. 2, string theory predicts in general that at tree level, the gauge couplings g_i corresponding to each gauge group factor G_i realized at Kač-Moody level k_i will unify with the gravitational coupling constant G_N according to [2]

$$g_{\text{string}}^2 = 8\pi \frac{G_N}{\alpha'} = k_i g_i^2 \quad \text{for all } i \quad (3.1)$$

where α' is the Regge slope. This unification occurs at the Planck scale. At the one-loop level, however, the string unification scale is shifted down to [4]

$$M_{\text{string}} \approx g_{\text{string}} \times 5 \times 10^{17} \text{ GeV} , \quad (3.2)$$

and the relations (3.1) are modified to

$$\frac{16\pi^2}{g_i^2(\mu)} = k_i \frac{16\pi^2}{g_{\text{string}}^2} + b_i \ln \frac{M_{\text{string}}^2}{\mu^2} + \Delta_i^{(\text{total})} \quad (3.3)$$

where b_i are the one-loop beta-function coefficients and where the numerical value quoted in (3.2) is computed in the \overline{DR} renormalization-group scheme. In (3.3), the

quantities $\Delta_i^{(\text{total})}$ represent the combined corrections from various string-theoretic and field-theoretic effects such as heavy string threshold corrections, light SUSY thresholds, intermediate gauge structure, and extra string-predicted matter beyond the MSSM. Thus, in a given realistic string model, we can use (3.3) to find the expected values of the strong and electroweak gauge couplings at the Z -scale, and thereby obtain explicit expressions for $\alpha_{\text{strong}}(M_Z)$ and $\sin^2 \theta_W(M_Z)$. Conversely, imposing the correct values of the low-energy couplings at the low energy scale, we can determine for which values of (k_Y, k_2, k_3) we obtain a consistent string-scale unification.

For the present analysis we shall ignore the contributions to the corrections $\Delta_i^{(\text{total})}$ which arise from such sources as heavy string thresholds, light SUSY thresholds, intermediate gauge structure, and extra non-MSSM matter. Of course, such corrections can be non-zero, and make contributions to the running of the couplings. Unfortunately, however, these corrections are also highly model-dependent, and therefore one cannot include them in such an analysis without making recourse to a particular string model. Indeed, within the framework of a large class of realistic string models, such a complete analysis has been performed in Refs. [5, 6], and while most of these corrections were found to be small (or, in general, insufficient to resolve the discrepancy between the MSSM unification scale and the string scale), one of these effects — namely that arising from string-predicted intermediate-scale matter beyond the MSSM — was found to be potentially large enough in certain models to resolve the discrepancies. Our present goal, however, is to determine the extent to which string-scale unification is consistent in a general setting *without* making recourse to such large corrections, but rather by judiciously choosing the Kač-Moody levels (k_Y, k_2, k_3) . In other words, our purpose here is to see to what extent a suitable foundation for string-scale unification can be established *before* such corrections are added.

We shall therefore disregard these correction terms, and thereby assume only the MSSM spectrum between the Z scale and MSSM scale. We can then solve the equations (3.3) simultaneously in order to remove the direct dependence on g_{string} . In this way, we find that the low-energy couplings $\sin^2 \theta_W(M_Z)$ and $\alpha_{\text{strong}}(M_Z)$ depend on the Kač-Moody levels (k_3, k_2, k_Y) as follows:

$$\begin{aligned} \sin^2 \theta_W(M_Z) &= \frac{1}{1+r} \left[1 - (b_Y - r b_2) \frac{a}{2\pi} \ln \frac{M_{\text{string}}}{M_Z} \right] \\ \alpha_{\text{strong}}^{-1}(M_Z) &= \frac{r'}{1+r} \left[\frac{1}{a} - \left(b_Y + b_2 - \frac{1+r}{r'} b_3 \right) \frac{1}{2\pi} \ln \frac{M_{\text{string}}}{M_Z} \right] \end{aligned} \quad (3.4)$$

where $a \equiv \alpha_{\text{e.m.}}(M_Z)$ is the electromagnetic coupling at the Z scale, and where

$$r \equiv \frac{k_Y}{k_2} \quad \text{and} \quad r' \equiv \frac{k_3}{k_2}. \quad (3.5)$$

Thus we see that the magnitude of $\sin^2 \theta_W$ depends on only the single ratio r , while the magnitude of α_{strong} depends on only the single ratio $r'/(1+r)$. These observations

imply that if we ignore the above corrections, the value of k_Y/k_2 can be determined purely by the value of the coupling $\sin^2 \theta_W$ at the Z scale, while the value of k_3/k_2 can then be adjusted in order to maintain an acceptable value for α_{strong} at the Z scale.

It is straightforward to determine the numerical constraints on these ratios. For the MSSM spectrum (three generations and two Higgs representations), we have $b_Y = 11$, $b_2 = 1$, and $b_3 = -3$; likewise the experimentally measured values for the couplings [25] at the Z scale are $\alpha_{\text{strong}}(M_Z)|_{\overline{MS}} = 0.120 \pm 0.010$ and $\sin^2 \theta_W(M_Z)|_{\overline{MS}} = 0.2315 \pm 0.001$. Note, however, that these values are determined in the \overline{MS} renormalization-group scheme, whereas the value for M_{string} in (3.2) is given in the \overline{DR} scheme. This means that although we are ignoring the possible effects from heavy string threshold corrections, light SUSY thresholds, intermediate gauge structure, and extra non-MSSM matter when we set $\Delta_i^{(\text{other})} = 0$ in (3.3), we should nevertheless retain the additional contributions to these quantities which come from renormalization-group scheme conversion. Likewise, there are also corrections from other model-independent sources such as two-loop effects, and the effects of minimal Yukawa couplings. These should also be included. We shall see, in fact, that these effects can be quite significant.

In order to estimate these effects, we shall proceed as follows. Let us denote by Δ_Y , Δ_2 , and Δ_3 the combined corrections to (3.3) from each of these sources. On the basis of previous calculations [5, 6], we expect that the two-loop corrections are significantly larger (by an order of magnitude) than those from the Yukawa couplings or scheme conversion, and we shall see this explicitly. To calculate the contributions to the Δ_i which come from two-loop effects, we evolve the MSSM couplings between the Z scale and the string scale with both the one-loop and Yukawa-less two-loop RGE's, assuming the MSSM spectrum with $(k_Y, k_2, k_3) = (5/3, 1, 1)$. We then take the difference in order to obtain the values of the two-loop corrections $\Delta_{Y,2,3}$. In this way, we find the numerical results

$$\Delta_Y^{(2\text{-loop})} \approx 15.5, \quad \Delta_2^{(2\text{-loop})} \approx 15.1, \quad \Delta_3^{(2\text{-loop})} \approx 7.8. \quad (3.6)$$

Moreover, as we shall demonstrate in Appendix B, these quantities depend only logarithmically on the ratios of the Kač-Moody levels k_i . We can therefore take these quantities to be fixed in our subsequent analysis. Likewise, to calculate the corrections from the Yukawa couplings, we evolve the two-loop RGE's for the gauge couplings coupled with the one-loop RGE's for the heaviest-generation Yukawa couplings, assuming $\lambda_t(M_Z) \approx 1.1$, $\lambda_b(M_Z) \approx 0.175$, and $\lambda_\tau(M_Z) \approx 0.1$. We then subtract the two-loop non-coupled result. This yields the following Yukawa-coupling corrections:

$$\Delta_Y^{(\text{Yuk})} \approx -3.9, \quad \Delta_2^{(\text{Yuk})} \approx -2.7, \quad \Delta_3^{(\text{Yuk})} \approx -1.8, \quad (3.7)$$

and we similarly expect the dependence of these quantities on the Kač-Moody levels k_i to be at most logarithmic. Finally, we can easily calculate the corrections of scheme

conversion between the \overline{DR} and \overline{MS} schemes, for these corrections are given directly in terms of the quadratic Casimirs of the different representations. We thus have

$$\Delta_Y^{(\text{conv})} = 0, \quad \Delta_2^{(\text{conv})} = 2/3, \quad \Delta_3^{(\text{conv})} = 1, \quad (3.8)$$

and these values are indeed strictly independent of the Kač-Moody levels k_i . Thus, combining the results (3.6), (3.7), and (3.8), we have the total corrections

$$\Delta_Y \approx 11.6, \quad \Delta_2 \approx 13.0, \quad \Delta_3 \approx 7.0. \quad (3.9)$$

We then find, including these corrections Δ_i into (3.3), that the two equations in (3.4) are likewise corrected by corresponding terms $\Delta^{(\text{sin})}$ and $\Delta^{(\alpha)}$ respectively added to their right sides, where

$$\begin{aligned} \Delta^{(\text{sin})} &= -\frac{1}{1+r} \frac{a}{4\pi} (\Delta_Y - r \Delta_2) \\ \Delta^{(\alpha)} &= -\frac{r'}{1+r} \frac{1}{4\pi} \left(\Delta_Y + \Delta_2 - \frac{1+r}{r'} \Delta_3 \right). \end{aligned} \quad (3.10)$$

Thus the primary level-dependence of these corrections is that given in (3.10).

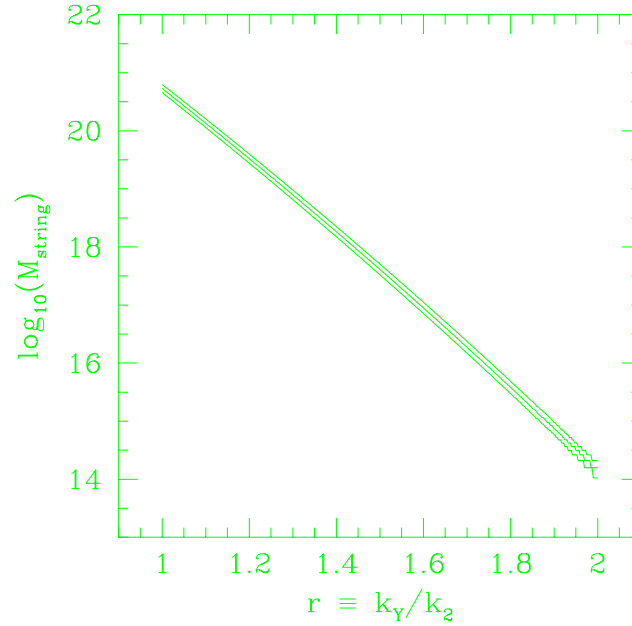


Figure 1: Dependence of unification scale M_{string} on the chosen value of $r \equiv k_Y/k_2$. The different curves correspond to different values of $\sin^2 \theta_W(M_Z)$, with the lower curve arising for greater values.

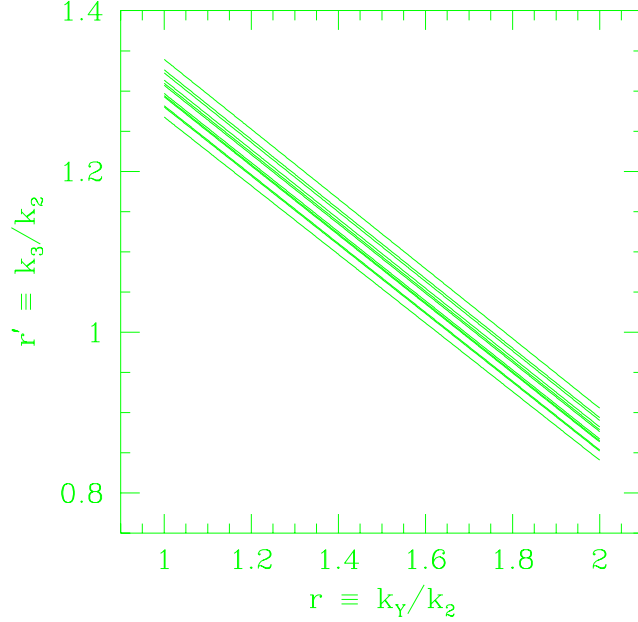


Figure 2: Values of $(r \equiv k_Y/k_2, r' \equiv k_3/k_2)$ yielding the experimentally observed low-energy couplings. The different curves correspond to different values of the couplings, with the higher curves arising for smaller values of $\sin^2 \theta_W(M_Z)$ and $\alpha_{\text{strong}}(M_Z)$, and the lower curves arising for greater values. Different points on any single curve correspond to different unification scales.

With these corrections included in the equations (3.4), we can now determine the experimental constraints on the ratios r and r' . Our results are shown in Figs. 1 and 2. In Fig. 1, we show the dependence of the scale of unification on the choice of the ratio r ; as expected, we find that $r = 5/3$ leads to a unification scale approximately at $M_{\text{MSSM}} \approx 2 \times 10^{16}$ GeV, while unification at the desired string scale $M_{\text{string}} \approx 5 \times 10^{17}$ GeV occurs only for smaller values of r , typically $r \approx 1.5$. Note that this curve relies on only the low-energy input from $a \equiv \alpha_{\text{e.m.}}(M_Z)$ and $\sin^2 \theta_W(M_Z)$. We have taken $a^{-1} \equiv 127.9$ as a fixed quantity, and have varied $\sin^2 \theta_W(M_Z)$ in the range $0.2305 \leq \sin^2 \theta_W(M_Z) \leq 0.2325$.

In Fig. 2, by contrast, we see how the value of r' must then be correspondingly adjusted in order to maintain an experimentally acceptable value for $\alpha_{\text{strong}}(M_Z)$. Thus, Fig. 2 summarizes those combinations of (r, r') which are consistent with the phenomenologically acceptable values for each of the three low-energy couplings, with $\sin^2 \theta_W(M_Z)$ allowed to vary in the above range, and with α_{strong} allowed to vary in the range $0.115 \leq \alpha_{\text{strong}} \leq 0.13$. It is clear from this figure that decreases in r must generally be accompanied by increases in r' in order to obtain acceptable low-energy couplings; moreover, only the approximate region $1.5 \leq r \leq 1.8$ is capable of yielding

$r' \approx 1$. This is an important constraint, for the Kač-Moody levels k_2 and k_3 are restricted to be *integers*, and thus arbitrary non-integer values of r' would generally be possible only for extremely high levels $k_2, k_3 \gg 1$.

The analysis thus far has only constrained the values of the *ratios* of the levels (k_Y, k_2, k_3) , for the renormalization group equations (3.4) give us no constraints concerning the *absolute sizes* of the Kač-Moody levels. Fortunately, however, there is one additional constraint which must be imposed in order to reflect the intrinsically stringy nature of the unification. In field theory, there are ordinarily two free parameters associated with unification: the value of the coupling at the unification scale, and the unification scale itself. In string theory, by contrast, these two parameters are tied together via (3.2). Thus, in string theory it is actually not sufficient to determine the ratios k_Y/k_2 and k_3/k_2 by merely demanding that they agree with low-energy data. Rather, we must also demand that if our low-energy couplings are run up to the unification point in a manner corresponding to certain values of the levels (k_Y, k_2, k_3) , the value of the predicted coupling g_{string} at the unification scale must be in agreement with the value of that unification scale. If this final constraint is not met, then we have not achieved a truly “stringy” unification.

This final constraint may be imposed as follows. Since $a \equiv \alpha_{\text{e.m.}}(M_Z)$ is the fixed “input” parameter of our analysis, it is most convenient to follow the running of $\alpha_{\text{e.m.}}$. Let us for the moment ignore the two-loop corrections Δ_i . Given the relation $1/\alpha_Y + 1/\alpha_2 = 1/\alpha_{\text{e.m.}}$ at all energy scales, we then find using (3.3) that

$$\frac{1}{\alpha_{\text{e.m.}}(\mu)} = \frac{1}{\alpha_{\text{e.m.}}(M_{\text{string}})} + \frac{b_Y + b_2}{4\pi} \ln \frac{M_{\text{string}}^2}{\mu^2}. \quad (3.11)$$

Note, however, that at the unification scale, we have

$$\alpha_{\text{string}}(M_{\text{string}}) = (k_Y + k_2) \alpha_{\text{e.m.}}(M_{\text{string}}). \quad (3.12)$$

Thus, taking $\mu = M_Z$ in (3.11) and substituting (3.12), we find

$$\frac{1}{\alpha_{\text{string}}(M_{\text{string}})} = \frac{1}{k_Y + k_2} \left[\frac{1}{a} - \frac{b_Y + b_2}{4\pi} \ln \frac{M_{\text{string}}^2}{M_Z^2} \right], \quad (3.13)$$

so that imposing (3.2) and defining $\alpha_{\text{str}} \equiv \alpha_{\text{string}}(M_{\text{string}})$ allows us to obtain the following transcendental equation for α_{str} :

$$\frac{1}{\alpha_{\text{str}}} = \frac{1}{k_Y + k_2} \left[\frac{1}{a} - \frac{b_Y + b_2}{2\pi} \ln \frac{\sqrt{4\pi\alpha_{\text{str}}} (5.27 \times 10^{17} \text{ GeV})}{M_Z (\text{GeV})} \right]. \quad (3.14)$$

Note that unlike the RGE’s in (3.4), this equation depends on the *absolute* sizes of the Kač-Moody levels k_i . Moreover, if we now restore the corrections Δ_i to (3.3), we find that the right side of this transcendental equation ultimately accrues a net correction

$$\Delta^{(\text{trans})} = - \frac{\Delta_Y + \Delta_2}{k_Y + k_2}. \quad (3.15)$$

This also depends on the absolute sizes of the levels k_Y and k_2 . Thus, assuming that the Δ_i are themselves essentially independent of the levels k_i , we can solve this corrected transcendental equation numerically for α_{str} as a function of $k_Y + k_2$. Then, by combining these results with our previous results restricting the allowed parameters with respect to low-energy couplings, we can actually fix not only the ratios k_Y/k_2 and k_3/k_2 of the Kač-Moody levels, but also their absolute values.

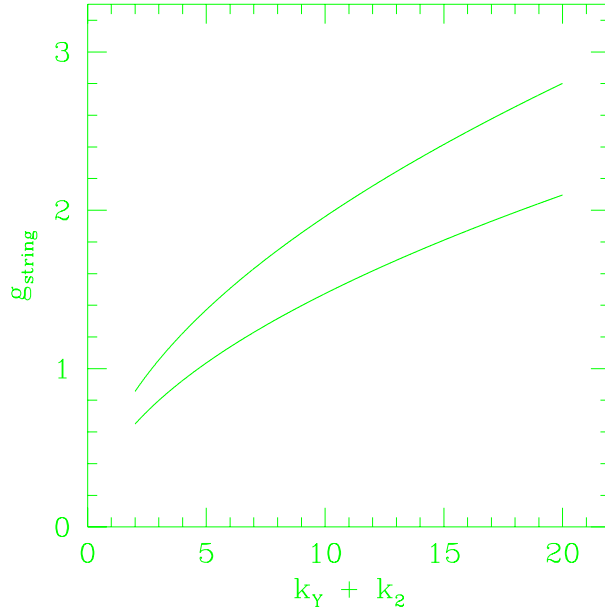


Figure 3: Solutions to the transcendental equation (3.14) for different values of $k_Y + k_2$, with the two-loop, Yukawa, and scheme-conversion corrections included (upper curve), and omitted (lower curve).

Our results are shown in Figs. 3 and 4. In Fig. 3, we display the numerical solutions to the transcendental self-consistency equation (3.14), plotting $g_{\text{string}} \equiv \sqrt{4\pi\alpha_{\text{str}}}$ as a function of $k_Y + k_2$, both with and without the two-loop, Yukawa, and scheme-conversion corrections. As we can see, the effect of the corrections (3.15) is quite significant. For $k_Y + k_2 = 5/3 + 1 = 8/3$, we find, as expected, that $0.7 \leq g_{\text{string}} \leq 1.0$, while for higher values of $k_Y + k_2$, the required string coupling increases significantly.

Combining this result with that in Fig. 1, we are finally able to correlate the acceptable values of the level ratio $r \equiv k_Y/k_2$ with the absolute sizes of the levels. Indeed, we see from Fig. 1 that the choice of a particular value of $r \equiv k_Y/k_2$ not only sets a particular unification scale, but with that choice of scale comes a certain scale for the string coupling via (3.2), and this in turn requires, via (3.14), a certain absolute magnitude for the Kač-Moody levels themselves. This combined dependence

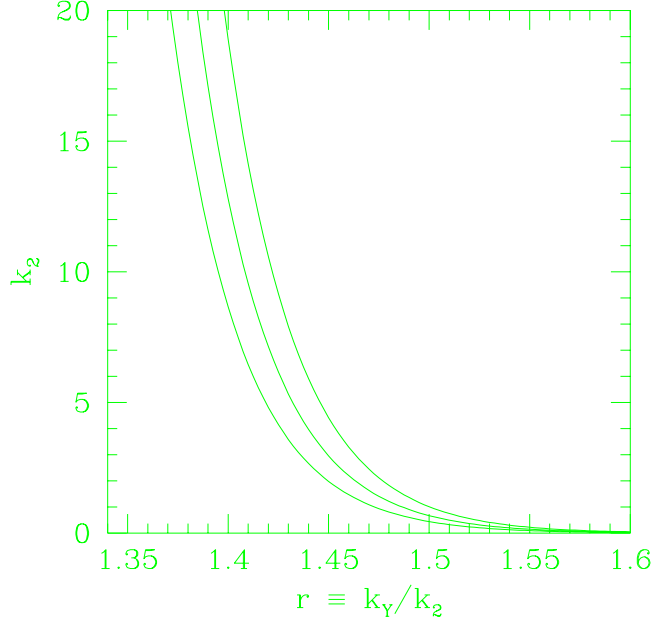


Figure 4: Dependence of k_2 on $r \equiv k_Y/k_2$. The absolute size of the Kač-Moody levels is set by the two-loop-corrected self-consistency constraint (3.14), in conjunction with the constraints from the low-energy value of $\sin^2 \theta_W(M_Z)$. The different curves correspond to different values of $\sin^2 \theta_W(M_Z)$, with the lower/left curves arising for greater values.

is shown in Fig. 4, and is clearly quite dramatic. In particular, we see from this figure that values of r in the range $1.45 \leq r \leq 1.5$ are consistent with relatively small Kač-Moody levels $k_{2,3} = 1, 2$, but that for smaller values of r , the required Kač-Moody levels increase dramatically. Interestingly, this figure also shows that there are natural *lower and upper bounds* on phenomenologically viable values of r if we wish to realize string-scale unification with only the MSSM spectrum. The upper bound arises from the fact that $k_2 \geq 1$, which implies that $r \leq 1.5$, or equivalently $k_Y \leq 1.5$. The lower bound, by contrast, arises if we wish to avoid unacceptably large values of g_{string} at unification. Specifically, Fig. 4 shows that if $r < 1.35$, then we are forced to realize this value of r through extremely large values of k_2 and k_Y , and from Fig. 3 we then see that such large values of k_2 and k_Y push the string coupling into the non-perturbative regime. Of course, we emphasize that the precise placement of the curves in Fig. 4 depends quite strongly on the exact value of $\sin^2 \theta_W(M_Z)$ (as shown), as well as on the various threshold corrections we have been neglecting. Indeed, although we expect these curves to remain essentially independent of these corrections above $r \approx 1.45$, we see that the region $r < 1.45$ is *exponentially* dependent on the precise corrections. Thus, these lower- r regions of the curves in Fig. 4 are best interpreted as qualitative

only. Nevertheless, the general shapes of these curves, evidently requiring extremely large changes in the absolute values of k_2 and k_3 for seemingly modest shifts in the ratio $r \equiv k_Y/k_2$, are striking and should provide extremely strong constraints on realistic string model-building.

Thus, combining all of these results, it is clear that only certain tightly-constrained values (k_Y, k_2, k_3) are phenomenologically allowed if we are to achieve string-scale unification with only the MSSM spectrum, and *without* resorting to large corrections from heavy string thresholds or extra non-MSSM matter. Indeed, from Figs. 2 and 4, we see that the best phenomenologically allowed regions of (k_Y, k_2, k_3) exist in a narrow band stretching through different values of r in the range $1.4 \leq r \leq 1.5$. At the lower end of this band, for example, we have values such as

$$r \approx 1.4 \quad \implies \quad (k_Y, k_2, k_3) \approx (21, 15, 17) , \quad (3.16)$$

whereas at intermediate regions of the band we have values such as

$$r \approx 1.42 \quad \implies \quad (k_Y, k_2, k_3) \approx (14.2, 10, 11) , \quad (3.17)$$

and at higher regions we have

$$r \approx 1.5 \quad \implies \quad (k_Y, k_2, k_3) \approx (1.5, 1, 1) \text{ or } (3, 2, 2) . \quad (3.18)$$

It is interesting to note that, strictly speaking, the *smaller* values of $r \approx 1.4$ are actually phenomenologically preferred, since they require the larger absolute values of (k_2, k_3) which in turn enable us to more precisely achieve the required corresponding values of r' shown in Fig. 2. This is an important observation, for the allowed region indicated in Fig. 2 encompasses two standard deviations in the low-energy couplings. Thus, any failure to achieve the precise required value of r' translates into a serious many-standard-deviation error in the low-energy couplings. Unfortunately, this low- r region is particularly sensitive to the corrections from, *e.g.*, heavy string threshold effects. Thus, it is difficult to make reliable model-independent statements concerning gauge-coupling unification in this region.

By contrast, for the purposes of *string model-building*, it turns out that smaller values for the non-abelian Kač-Moody levels, such as $k_2, k_3 = 1$ or 2 , are strongly preferred on practical grounds. Indeed, not only does it grow increasingly difficult to build string models with higher values of (k_2, k_3) , but as these non-abelian levels are increased, unwanted $SU(3)$ and $SU(2)$ representations of increasing dimensionality begin to appear in the massless spectrum. Thus, for the purposes of string model-building, we find that the preferred regions of parameter space are actually in the *higher- r* region, with $r \approx 1.5$, and with $k_2 = k_3 = 1$ or 2 . We would then hope that relatively *small* corrections from other sources such heavy string thresholds would increase the effective value of r' from 1 to $r' \approx 1.05 - 1.1$. Furthermore, in this higher- r region, we do not expect such corrections to have a major effect on the

validity of the curves in Fig 4. We shall therefore focus our attention on the values

$$\begin{cases} k_2 = k_3 = 1 \text{ or } 2 \\ r \equiv k_Y/k_2 \approx 1.45 - 1.5 \end{cases} \quad (3.19)$$

in what follows.

4 Hypercharge Embeddings

We now turn to the central question that we seek to address in this paper: within the set of self-consistent “realistic” string models containing the MSSM gauge group and particle representations (as listed in Appendix A), how might models be constructed with levels (k_Y, k_2, k_3) in the phenomenologically preferred region (3.19)? In particular, for realistic string models with levels $k_2 = k_3 = 1$ or 2 , what are the allowed values for the hypercharge normalization k_Y ? There are essentially two “orthogonal” approaches that we will follow. In the next two sections, we shall focus on possible string embeddings of the hypercharge group factor. In this way, by imposing certain constraints, we will arrive at a relatively small number of possible values of $k_Y < 5/3$. Indeed, one of the main results of this section will be a proof that within a certain broad class of realistic level-one string models, one must always have $k_Y \geq 5/3$. In Sect. 6, by contrast, we shall instead focus on the implications of the phenomenological requirement of charge integrality for string states. As we shall see, this provides an alternative method of obtaining general constraints on allowed values of (k_Y, k_2, k_3) .

4.1 Hypercharge Embeddings: General Strategy

As discussed in Sect. 2, the value of k_Y in a given string model is determined by the manner in which its hypercharge group factor $U(1)_Y$ is ultimately embedded within the given elementary $U(1)$ factors. Therefore, in this section, we shall focus on the kinds of hypercharge embeddings that are allowed in string theory and that can potentially yield all of the the MSSM representations with consistent hypercharge assignments.

There are various strengths and weaknesses to this approach. On the one hand, by focusing on stringy *embeddings* rather than on any string *models* themselves, our analysis is in some sense independent of any particular string construction. Thus, if we are able to prove that no suitable *embeddings* with $k_Y < 5/3$ exist (as we shall be able to do for a certain class of string models), this result will therefore hold for all string models in this class, whether constructed via free fermions, free bosons, *etc.* On the other hand, this approach does suffer from certain weaknesses. Since an MSSM embedding is ultimately only a small part of a given string model, an analysis of an MSSM embedding alone is often beyond the reach of certain powerful *string* consistency constraints such as modular invariance. Indeed, the string charge lattice

is generally of dimension 22, whereas the MSSM group is typically embedded in only the first few dimensions of this lattice. Without knowledge of what is happening in the remaining dimensions of this lattice, we cannot impose the lattice constraints (such as self-duality) that arise from modular invariance. Therefore, if we are able to construct interesting MSSM embeddings with $k_Y < 5/3$ (as we will be able to do for certain other classes of string theories), the question will still remain as to whether there actually exist string *models* which realize these embeddings in a manner consistent with modular invariance and other string consistency conditions.

By contrast, the alternative approach we shall follow in Sect. 6 will incorporate modular invariance at an early stage. Unfortunately, however, its power will be that of an existence proof: it will be capable of determining under which conditions string models with $k_Y < 5/3$ might exist, but it will offer no clues as to their constructions or embeddings. Thus, we follow both approaches in the hope that some judicious mixture of the two will be most fruitful in the long run.

Our analysis of potential hypercharge embeddings will proceed in several stages. First, we will examine the possible string embeddings of the non-abelian factors of the MSSM gauge group, namely $SU(3)_C$ and $SU(2)_L$. This will ultimately allow us to identify the potential string embeddings of the MSSM spectrum (*i.e.*, the realization of the MSSM representations as excitations of underlying worldsheet fields), which will in turn restrict their quantum numbers under the constituent worldsheet $U(1)$ factors. Given this information, we will then be able to examine to survey possible classes of embeddings which enable a consistent hypercharge assignment to be realized. This will then enable us to deduce their corresponding values of k_Y .

4.2 Charge Lattices: Basic Facts

The most efficient method of discussing a particular string-theoretic embedding is by specifying its associated *charge lattice*. This may be defined as follows. *A priori*, a heterotic string theory can give rise to gauge groups of maximal rank 22, and correspondingly there exists a maximal set of 22 Cartan generators H_i , $i = 1, \dots, 22$. These generators are of course the elementary $U(1)$ currents we discussed in Sect. 2; for example, in a free-fermionic model whose internal sector is built out of 22 complex worldsheet fermions f_i , these currents are simply $H_i \equiv f_i^* f_i$. Equivalent identifications may also be made for other constructions. Indeed, in some cases (such as those involving necessarily real fermions), the number of such Cartan generators surviving the GSO constraints may actually be smaller than 22 (leading to the phenomenon of *rank-cutting*, whereby the total rank of the resulting gauge group is less than 22). In any case, however, given a set of r different Cartan generators H_i , each state in the particle spectrum can then be described by specifying its eigenvalues or *charges* Q_i with respect to these generators. The set of such r -dimensional vectors \mathbf{Q} corresponding to all of the states in the model then fills out the “charge lattice” of the model, and from this information the particle representations and gauge group can be

determined. For example, the set of charge vectors corresponding to the gauge bosons in the string spectrum fills out the root lattice of the corresponding gauge group, and similarly the charge vectors corresponding to the particles in a given multiplet fills out the weight system of the corresponding representation. Since each direction of the charge lattice corresponds to one of the underlying worldsheet $U(1)$ currents, we therefore see that specifying the charge vector of a particular state uniquely specifies its embedding, or equivalently its realization in terms of excitations of the underlying worldsheet fields.*

As we have already remarked, working with the charge lattice of states is particularly convenient because it provides a method of analysis which is largely independent of the particular string model-building method of construction. Indeed, the existence of a spacetime gauge group of rank r implies the existence of such an r -dimensional charge lattice, and the entire string spectrum must therefore fill out complete representations with respect to this lattice. Moreover, certain properties of this lattice can be deduced without reference to the particular underlying string construction. For example, modular invariance tightly constrains this lattice, requiring that when it is tensored together with a similar lattice from the supersymmetric side of the heterotic string, the resulting product lattice must be *self-dual*. Likewise, the requirement that the underlying worldsheet conformal field theories be rational implies that every charge vector must have *rational components*. Furthermore, even the *lengths* of the charge vectors are tightly constrained. For example, the squared lengths of the charge vectors \mathbf{Q} corresponding to the non-Cartan gauge bosons of any simply-laced group G realized at level k_G are constrained (in the normalizations conventional to string theory) to be

$$\mathbf{Q} \cdot \mathbf{Q} = \frac{2}{k_G}, \quad (4.1)$$

and for non-simply laced groups this result holds for the gauge bosons corresponding to the long roots. As we shall see, all of these facts will prove crucial in our analysis.

4.3 Gauge Group Embeddings

We now turn to the possible string embeddings of the non-abelian factors of the MSSM gauge group, namely $SU(3)_C$ and $SU(2)_L$. We shall assume, for the purposes of this analysis, that these non-abelian group factors are realized at Kač-Moody levels $k_2 = k_3 = 1$.

We begin with the embedding of the $SU(2)$ factor. As we have seen in Sect. 2, such a group factor has central charge $c = 1$, and likewise has rank $r = 1$. Now, in general each dimension of the charge lattice corresponds to central charge $c = 1$; this is true for all models built from worldsheet bosons (regardless of their method of

* Strictly speaking, this statement is true only for massless states; otherwise, one must specify both the charge vector and the energies of excitation. However, for our present purposes, we will only be considering the massless (*i.e.*, observable) states that comprise the MSSM spectrum.

compactification), or worldsheet fermions (complex or real). Indeed, this is true for all of the free-field model-construction procedures through which phenomenologically appealing realistic string models have been constructed. Consequently, we see that an $SU(2)$ gauge group factor at level $k_2 = 1$ must be realized through a lattice of at least dimensionality one. However, we see from (4.1) that for level $k_2 = 1$, a one-dimensional lattice would require the non-Cartan $SU(2)$ gauge boson to have a charge of the form $\mathbf{Q} = (\sqrt{2})$, which fails to contain rational components. Thus, we see that $SU(2)$ at level one cannot be realized completely within a single dimension of the charge lattice, and must be realized instead as a non-trivial embedding across several dimensions. In this case, however, it is easy to see that there *does* exist a suitable *two-dimensional* embedding, with non-Cartan charges

$$(-1, 1) \quad \text{and} \quad (1, -1) . \quad (4.2)$$

[By changing the relative orientation of these dimensions, this is equivalent to $(1, 1)$ and $(-1, -1)$, but the above orientation is chosen for future convenience.] As required, these charge vectors have only rational components. Thus, we see that the minimal embedding of the $SU(2)_1$ root lattice is one in which the $SU(2)$ axis lies along the diagonal between *two* dimensions of the total charge lattice. The orthogonal direction $(1, 1)$ can then correspond to an additional $SU(2)$ factor, or a $U(1)$ factor. We will denote the “unit” charge vector in this orthogonal direction as $\mathbf{Q}_L \equiv (1, 1)$.

A similar analysis can be performed in order to determine the minimal $SU(3)$ embedding. Since $SU(3)_1$ has central charge and rank equal to two, *a priori* only two dimensions of the charge lattice are needed. However, as expected, no such embedding of the $SU(3)$ root system with rational components exists, and instead three dimensions of the charge lattice are required. The three-dimensional embeddings for the two simple roots of the $SU(3)$ root system are then

$$(-1, 1, 0) \quad \text{and} \quad (0, -1, 1) , \quad (4.3)$$

and it is easy to verify that these two vectors have a relative angle of 120° , as required. The remaining positive non-Cartan generator for $SU(3)_1$ is then the sum of these roots, or $(-1, 0, 1)$. As in the $SU(2)$ case, the direction perpendicular to the $SU(3)$ plane corresponds to an additional $U(1)$ factor, and we shall choose a “unit” charge vector in this orthogonal direction to be $\mathbf{Q}_C \equiv (1, 1, 1)$. Note that in this $SU(3)$ case, the extra orthogonal direction cannot correspond to an $SU(2)$ factor, since \mathbf{Q}_C does not lie diagonally between two directions of the charge lattice [*i.e.*, it is not of the form $(1, 1, 0)$ or permutations thereof].

These results imply, therefore, that the minimal embedding of the gauge group $SU(3) \times SU(2)$ at levels $k_3 = k_2 = 1$ in string theory requires a *five-dimensional* lattice. We shall henceforth take these to be the first five directions of the charge lattice, with eigenvalues Q_1, Q_2, Q_3 pertaining to the $SU(3)$ embedding, and eigenvalues Q_4, Q_5 pertaining to the $SU(2)$ embedding. As discussed above, we then have

$Q_C \equiv Q_1 + Q_2 + Q_3$ and $Q_L \equiv Q_4 + Q_5$. We emphasize, however, that this particular embedding is only the simplest (and most compact) embedding for the $SU(3)_C$ and $SU(2)_L$ group factors realized at levels $k_2 = k_3 = 1$. There are, of course, many other potential embeddings for these gauge group factors which involve many more than five components of the charge lattice. This embedding, however, is “minimal” in that it involves the fewest number of elementary dimensions of the charge lattice, and is therefore most likely to yield the smallest values of k_Y .

4.4 Embeddings for matter representations

Given the above embeddings for the root systems (*i.e.*, for the adjoint representations), it is now straightforward to determine the possible embeddings for the matter representations that concern us, namely the **2** of $SU(2)$, and the **3** and $\bar{\mathbf{3}}$ of $SU(3)$. Since the lengths of the weight vectors of the doublet representation of $SU(2)$ are half those for the adjoint, we immediately see that the doublet of $SU(2)_1$ is realized as:

$$\mathbf{2} \text{ of } SU(2) : \quad \pm (-1/2, +1/2) . \quad (4.4)$$

Likewise, since the squared lengths of the weight vectors of the triplet representation of $SU(3)$ are one third those for the adjoint, we immediately see that given the above embedding for the $SU(3)$ roots, we have the following embedding for the $SU(3)$ triplet:

$$\mathbf{3} \text{ of } SU(3) : \quad \begin{cases} (-1/3, -1/3, +2/3) \\ (-1/3, +2/3, -1/3) \\ (+2/3, -1/3, -1/3) . \end{cases} \quad (4.5)$$

The $\bar{\mathbf{3}}$ representation of $SU(3)$ is of course given by the negatives of these three weights.

In constructions of string models based on worldsheet bosonic degrees of freedom, the denominators of the components of the charge lattice are related to the bosonic radii of compactification. Likewise, in a free-fermionic construction of string models, these denominators are related to the boundary conditions of the worldsheet fermions as they traverse the non-contractible cycles of the torus. Indeed, in constructions employing only periodic (Ramond) or anti-periodic (Neveu-Schwarz) fermions, the greatest denominator that the charge lattice can have is 2. This does not mean, however, that $SU(3)$ triplets cannot be obtained in such models; rather, this observation implies that if such $SU(3)$ triplets do appear, they must also carry a non-zero quantum number under the orthogonal $U(1)_C$ generator $Q_C \equiv (1, 1, 1)$. For example, we see that the $SU(3)$ triplet with $U(1)_C$ quantum number $1/3$ corresponds to charge vectors of the form $(0, 0, 1)$ and permutations. Such states with integer charge components are typically obtained from Neveu-Schwarz sectors. Similarly, the triplet with $U(1)_C$ quantum number $-1/6$ will have charge vectors of the form $(-1/2, -1/2, +1/2)$, which will typically be obtained in a Ramond sector. However, we see from the general considerations above that such non-zero $U(1)_C$ quantum

numbers are *required* if we are using only periodic or anti-periodic fermions and the minimal $SU(3)$ embedding.

Thus, in general, we see that we must allow our $SU(3)$ and $SU(2)$ representations to simultaneously carry charges under $U(1)_C$ and $U(1)_L$ respectively. Since these $U(1)$ gauge factors correspond to the $(1, 1, 1)$ and $(1, 1)$ directions respectively, we see that if the $SU(2)$ doublet has $U(1)_L$ charge q_L and the $SU(3)$ triplet has $U(1)_C$ charge q_C , then these representations correspond to the charge vectors:

$$\begin{aligned} (2)_{q_L} \text{ of } SU(2) : & \quad \pm(-1/2, +1/2) + q_L(1, 1) \\ (3)_{q_C} \text{ of } SU(3) : & \quad \left\{ \begin{array}{l} (-1/3, -1/3, +2/3) \\ (-1/3, +2/3, -1/3) \\ (+2/3, -1/3, -1/3) \end{array} \right\} + q_C(1, 1, 1) . \end{aligned} \quad (4.6)$$

Likewise, since the uncharged singlet representations correspond to $(0, 0, 0)$ and $(0, 0)$, the corresponding charged representations correspond to $q_C(1, 1, 1)$ and $q_L(1, 1)$ directly:

$$\begin{aligned} (1)_{q_L} \text{ of } SU(2) : & \quad q_L(1, 1) \\ (1)_{q_C} \text{ of } SU(3) : & \quad q_C(1, 1, 1) . \end{aligned} \quad (4.7)$$

4.5 Conformal Dimensions

Given these representations of $SU(3)$ and $SU(2)$, our next step is to examine their conformal dimensions. This is relevant because massless string states must have total conformal dimensions equal to 1, and this in turn will constrain the allowed values for the $U(1)_C$ and $U(1)_L$ quantum numbers that such representations can carry. For string models without rank-cutting (*i.e.*, models with rank-22 simply-laced gauge groups), the conformal dimension $h(\mathbf{Q})$ of a given state can in general be determined from its charge vector \mathbf{Q} via the relationship

$$h(\mathbf{Q}) = \frac{1}{2} \mathbf{Q} \cdot \mathbf{Q} . \quad (4.8)$$

Indeed, this relationship holds for all states except those adjoint gauge boson states in the Cartan subalgebra. Thus, calculating the conformal dimensions of the representations in (4.6) and (4.7), we have

$$\begin{aligned} SU(3) \times U(1)_C : & \quad \left\{ \begin{array}{ll} (1)_{q_C} : & h = (3/2) q_C^2 \\ (3)_{q_C} : & h = 1/3 + (3/2) q_C^2 \end{array} \right. \\ SU(2) \times U(1)_L : & \quad \left\{ \begin{array}{ll} (1)_{q_L} : & h = q_L^2 \\ (2)_{q_L} : & h = 1/4 + q_L^2 \end{array} \right. . \end{aligned} \quad (4.9)$$

Note that these results could also have been obtained directly from (2.5).

There are certain immediate lessons that can be drawn from these results. A trivial result, for example, is that there are natural bounds on the values of q_C

and q_L that each representation can carry while still remaining massless. A more profound result, however, is the observation that in string theory, we cannot realize all of our MSSM representations as singlets under additional (*i.e.*, hidden) gauge group factors in the string model. Rather, we are *forced* to have at least some of our MSSM representations transform non-trivially under some other hidden factors beyond $SU(3) \times SU(2) \times U(1)_C \times U(1)_L$. This is most easily seen by examining the lepton $L \equiv (\mathbf{1}, \mathbf{2})$ representation. If this state were to be realized as a singlet under all non-MSSM gauge factors, then the contribution to its conformal dimension from the MSSM factors would itself be equal to one, or

$$\frac{3}{2} q_C^2 + q_L^2 \stackrel{?}{=} \frac{3}{4} . \quad (4.10)$$

Remarkably, however, this equation has no solution for rational values of (q_C, q_L) . Hence, some of the total conformal dimension of this representation must be attributed to string excitations beyond the degrees of freedom giving rise to the MSSM factors. In general, all of the MSSM representations will have some non-MSSM conformal dimensions as well. Hence the extra (hidden) gauge group factors play a non-trivial role in realizing the MSSM states as massless excitations in string theory.

4.6 Chirality

We now discuss the implications of the fact that our representations (A.1) are to be *chiral*. This means, of course, that while we not only demand that the representations listed in (A.1) actually appear, we must also simultaneously demand that their complex conjugates *not* appear. In terms of the charge lattice of vectors \mathbf{Q} , this chirality constraint means the following. In general, the full charge lattice of a given string theory is a Lorentzian lattice of dimension (r', r) where the first $r' \leq 10$ dimensions correspond to the right-movers (the supersymmetric side of the heterotic string), and the remaining $r \leq 22$ dimensions correspond to the left-movers (the internal bosonic side of the heterotic string). The first of the right-moving charge components corresponds to the spacetime statistics of a given state (integer for bosons, integer $+1/2$ for fermions), and the first five left-moving components are simply the five-dimensional charge vectors \mathbf{Q} that we have been discussing above. Now, CPT invariance of the string spectrum implies that for every (r', r) -dimensional charge vector \mathbf{Q} that appears in the string spectrum, there must also appear the charge vector $-\mathbf{Q}$. Chirality, by contrast, involves all but the first (spacetime) component of the charge vector, and implies that for any given charge vector $\mathbf{Q} = (Q_1, Q_2, \dots, Q_{r'+r})$, the chiral conjugate $\tilde{\mathbf{Q}} \equiv (+Q_1, -Q_2, \dots, -Q_{r'+r})$ must not appear.

In general, this is a difficult requirement to implement in a general fashion, and whether a particular representation appears chirally or non-chirally usually depends on the intricate details of the GSO projections in the string model in question. One general fact that we will use, however, is that any string state whose internal charge vector \mathbf{Q} is of the form $(\pm 1, \pm 1, 0, 0, \dots, 0)$ or permutations cannot be chiral. This

observation follows from the fact that such a left-moving charge vector is also the left-moving charge vector of a gauge boson in the theory, and such gauge bosons (which belong to the adjoint) necessarily are non-chiral. This assertion holds whether this hypothetical gauge boson state is actually in the string spectrum, or is GSO-projected out (*i.e.*, whether the gauge group to which it corresponds is preserved, or broken). In either case such a state will either appear with its conjugate, or not appear at all. Thus, the chirality of our MSSM representations demands that no MSSM matter state can have the form

$$\mathbf{Q}_{\text{non-chiral}} = (\pm 1, \pm 1, 0, 0, 0, \dots 0) \quad \text{or permutations} . \quad (4.11)$$

4.7 Hypercharge Embeddings

Given the above minimal embeddings for the $SU(2)$ and $SU(3)$ adjoint and fundamental representations, we can now put the pieces together to examine the potential hypercharge embeddings which successfully reproduce the required hypercharge assignments.

As we have seen above, in string theory we cannot assume that the MSSM representations are charged under only the MSSM gauge group; rather, we are forced to have them transform non-trivially under the additional (*i.e.*, hidden) gauge group factors in the string model. Thus, the MSSM representations will in general have non-zero quantum numbers (*i.e.*, non-zero charge components Q_i) beyond the first five Cartan generators corresponding to the $SU(3) \times SU(2)$ embedding.

Despite this fact, we will restrict the present analysis to weak hypercharges Y that take the “minimal” form

$$Y \equiv \sum_{i=1}^5 a_i Q_i . \quad (4.12)$$

This restriction implies, of course, that Y is independent of any extra hidden gauge group factors. Note that this does not contradict our assertion that the MSSM representations must be charged under such factors, for it is possible that the *weak hypercharge* does not depend on these extra factors. Indeed, many of the realistic string models which exist in the literature have hypercharge embeddings of the form (4.12). Furthermore, since this is in some sense the most “compact” distribution of the hypercharge — involving the MSSM components of the lattice only — we may expect that this form is likely to yield the smallest values of k_Y . In any case, since we have no model-independent information regarding the remaining “hidden” components of the charge lattice (other than self-duality of the *entire* 32-dimensional left/right charge lattice), it is clear that we are unable to perform any detailed model-independent analysis for hypercharge embeddings beyond those of the minimal form (4.12).

Given the form (4.12), then, our goal is to determine the possible solutions for the a_i coefficients. The corresponding value of k_Y is then determined via (2.14).

The first thing we notice is that since the value of Y must be the same for all members of a given MSSM representation, Y must have the same eigenvalue for all members of an $SU(3)$ triplet or $SU(2)$ doublet. This immediately implies that

$$a_1 = a_2 = a_3 \quad \text{and} \quad a_4 = a_5 . \quad (4.13)$$

Let us denote $a_1 = a_2 = a_3 \equiv A_1$ and $a_4 = a_5 \equiv A_2$. Then acting on *any* $SU(3) \times SU(2)$ representation, we simply find

$$Y(\mathbf{R}_{q_C}, \mathbf{R}'_{q_L}) = 3A_1 q_C + 2A_2 q_L . \quad (4.14)$$

Hence, only the q_C and q_L eigenvalues are relevant for Y . In order to solve for A_1 and A_2 , therefore, we must first determine the q_C and q_L quantum numbers of each MSSM representation.

In general, there are many possibilities for the charges (q_C, q_L) for each of the MSSM representations (A.1), for these charges depend on the details of the underlying string model. Likewise, for each different combination, there may or may not exist a solution (A_1, A_2) which describes a corresponding successful hypercharge embedding. Hence, we must first narrow down the list of (q_C, q_L) possibilities. There are a variety of means through which this can be done. First, we recall of course that the rationality of the worldsheet conformal field theory requires that these charges be rational numbers. Second, we see from (4.9) that for our MSSM states to be potentially massless, the (q_C, q_L) quantum numbers of each $SU(3) \times SU(2)$ representation must satisfy the constraints:

$$\begin{aligned} (\mathbf{3}, \mathbf{2}) : & \quad (3/2) q_C^2 + q_L^2 \leq 5/12 \\ (\bar{\mathbf{3}}, \mathbf{1}) : & \quad (3/2) q_C^2 + q_L^2 \leq 2/3 \\ (\mathbf{1}, \mathbf{2}) : & \quad (3/2) q_C^2 + q_L^2 \leq 3/4 \\ (\mathbf{1}, \mathbf{1}) : & \quad (3/2) q_C^2 + q_L^2 \leq 1 . \end{aligned} \quad (4.15)$$

Third, in order for the representation $(\mathbf{R}_{q_C}, \mathbf{R}'_{q_L})$ to be potentially chiral, the corresponding five-dimensional charge vector $\mathbf{Q} = (Q_1, Q_2, Q_3, Q_4, Q_5)$ must not be of the form (4.11). Given the results in (4.6) and (4.7), this too rules out certain combinations of (q_C, q_L) .

Finally, we must impose a *moding requirement* on the charge vectors \mathbf{Q} corresponding to each representation. In general, this means that given a certain moding for the underlying worldsheet fields of the string, there exists a certain corresponding value for the greatest common denominator Δ of all of the elements Q_i of the charge vectors of the corresponding spacetime states. For example, if a string model is built in the free-fermionic construction using fermions with only Ramond or Neveu-Schwarz (periodic or anti-periodic) boundary conditions around the non-contractible cycles of the torus, then we have $\Delta = 2$, and all of the allowed states in such a model must have charge vectors whose components are integers or half-integers only. In general Δ must be a positive even integer.

Since the analysis of all possible hypercharge configurations is highly dependent on the particular value of Δ chosen, we must examine each case in turn. For each value of Δ , we shall therefore simply enumerate all possible values of (q_C, q_L) for each MSSM representation subject to the masslessness, chirality, and moding constraints, and for each set of combinations of (q_C, q_L) we will then determine whether a consistent hypercharge assignment of the form (4.12) exists. If so, we can then quickly determine the corresponding value of k_Y which describes the particular embedding, and thereby determine the minimum value of k_Y obtained.

We begin with the simplest case $\Delta = 2$, and defer a discussion of the higher-twist cases with $\Delta > 2$ to the next section. Given our masslessness, chirality, and moding restrictions, we then find that for $\Delta = 2$, the following are the only allowed combinations for (q_C, q_L) :

$$\begin{aligned} (\mathbf{3}, \mathbf{2}) : \quad & (q_C, q_L) = (1/3, 0), (-1/6, 0), (-1/6, \pm 1/2) \\ (\bar{\mathbf{3}}, \mathbf{1}) : \quad & (q_C, q_L) = (-1/3, 0), (-1/3, \pm 1/2), (1/6, 0), (1/6, \pm 1/2) \\ (\mathbf{1}, \mathbf{2}) \text{ and } (\mathbf{1}, \mathbf{1}) : \quad & (q_C, q_L) = (0, \pm 1/2), (\pm 1/2, 0), \pm(1/2, \pm 1/2) . \end{aligned} \quad (4.16)$$

Thus, if we restrict our attention to the non-Higgs representations, we see that we have in principle $4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 = 9216$ different possible combinations of (q_C, q_L) charges for the five MSSM representations $\{Q, u, d, L, e\}$. It is straightforward to automate an examination of each of these possibilities, and indeed one finds that only for 16 combinations can a consistent hypercharge Y of the form (4.12) be defined. In fact, many of these 16 are related to each other through trivial changes of sign, so that there are only three different values of hypercharge normalization k_Y obtained. These are

$$k_Y = 5/3, 11/3, \text{ and } 14/3 , \quad (4.17)$$

with linear-combination coefficients $(A_1, A_2) = (-1/3, \pm 1/2)$, $(2/3, \pm 1/2)$, and $(-1/3, \pm 1)$ respectively. Thus we conclude that for $\Delta = 2$ case and with Y of the form (4.12), we have

$$k_Y \geq 5/3 . \quad (4.18)$$

Because of their importance, we now list the minimal embeddings with $k_Y = 5/3$. It turns out that, modulo sign flips and permutations, there are essentially two such inequivalent embeddings with $k_Y = 5/3$. The first is the standard “ $SO(10)$ embedding”, so called due to the fact that the charge vectors of the MSSM matter representations fill out the weights of the **16** representation of $SO(10)$.[†] In this embedding, the (q_C, q_L) charges and a typical \mathbf{Q} vector for each of the MSSM representations are as follows:

$$\begin{array}{ll} Q : & (q_C, q_L) = (-1/6, 0) \quad \mathbf{Q} = (-1/2, -1/2, +1/2, -1/2, +1/2) \\ u : & (q_C, q_L) = (+1/6, -1/2) \quad \mathbf{Q} = (1/2, 1/2, -1/2, -1/2, -1/2) \end{array}$$

[†] This includes the right-handed neutrino N representation, which we have not listed in (4.19).

$$\begin{aligned}
d : \quad (q_C, q_L) &= (+1/6, +1/2) & \mathbf{Q} &= (1/2, 1/2, -1/2, +1/2, +1/2) \\
L : \quad (q_C, q_L) &= (+1/2, 0) & \mathbf{Q} &= (+1/2, +1/2, +1/2, -1/2, +1/2) \\
e : \quad (q_C, q_L) &= (-1/2, +1/2) & \mathbf{Q} &= (-1/2, -1/2, -1/2, +1/2, +1/2) . \quad (4.19)
\end{aligned}$$

This embedding corresponds to the solution $(A_1, A_2) = (-1/3, +1/2)$, which successfully reproduces the required hypercharge assignments (A.5).

The second embedding with $k_Y = 5/3$ bears no relation to the $SO(10)$ embedding, yet shares the same value of k_Y . In this embedding, the (q_C, q_L) charges and a typical \mathbf{Q} vector for each of the MSSM representations are as follows:

$$\begin{aligned}
Q : \quad (q_C, q_L) &= (-1/6, 0) & \mathbf{Q} &= (-1/2, -1/2, +1/2, -1/2, 1/2) \\
u : \quad (q_C, q_L) &= (+1/6, -1/2) & \mathbf{Q} &= (1/2, 1/2, -1/2, -1/2, -1/2) \\
d : \quad (q_C, q_L) &= (-1/3, 0) & \mathbf{Q} &= (0, 0, -1, 0, 0) \\
L : \quad (q_C, q_L) &= (0, -1/2) & \mathbf{Q} &= (0, 0, 0, -1, 0) \\
e : \quad (q_C, q_L) &= (-1/2, +1/2) & \mathbf{Q} &= (-1/2, -1/2, -1/2, 1/2, 1/2) . \quad (4.20)
\end{aligned}$$

This embedding also corresponds to the solution $(A_1, A_2) = (-1/3, +1/2)$, which successfully reproduces the required hypercharge assignments (A.5) with these charge assignments. It is a trivial matter to verify that in both of these embeddings with $k_Y = 5/3$, the Higgs representations can also be successfully accommodated [*e.g.*, with charges $(q_C, q_L)_{H^\pm} = (0, \pm 1/2)$].

It is an interesting observation that two independent embeddings both have the “minimal value” $k_Y = 5/3$, and that all other possible embeddings with $\Delta = 2$ have only larger values of k_Y . In this regard, we emphasize that our chirality constraint has played a large role in achieving this result. For example, there *a priori* exists an alternative embedding which corresponds to an even smaller value of k_Y :

$$\begin{aligned}
Q : \quad (q_C, q_L) &= (+1/3, 0) & \mathbf{Q} &= (0, 0, 1, -1/2, 1/2) \\
u : \quad (q_C, q_L) &= (-1/3, +1/2) & \mathbf{Q} &= (0, 0, -1, 1/2, 1/2) \\
d : \quad (q_C, q_L) &= (-1/3, -1/2) & \mathbf{Q} &= (0, 0, -1, -1/2, -1/2) \\
L : \quad (q_C, q_L) &= (0, +1/2) & \mathbf{Q} &= (0, 0, 0, 0, 1) \\
e : \quad (q_C, q_L) &= (0, -1) & \mathbf{Q} &= (0, 0, 0, -1, -1) . \quad (4.21)
\end{aligned}$$

For this set of (q_C, q_L) charges, a consistent hypercharge Y can be found corresponding to the solution $(A_1, A_2) = (1/6, -1/2)$. This would lead to a hypercharge normalization $k_Y = 7/6 \approx 1.17$, which is substantially smaller than $5/3 \approx 1.67$. However, we see from (4.21) that in this embedding, the positron has the same charge vector as a gauge boson state (in particular, the gauge boson perpendicular to $SU(2)_L$). Hence this representation cannot be chiral: either it is projected out of the string spectrum altogether, or it appears with its complex conjugate. It is for this reason that the charge combination $(q_C, q_L) = (0, \pm 1)$ is not listed as a possibility for the $(\mathbf{1}, \mathbf{1})$ singlet representation in (4.16). The only other combination that was eliminated for

the same reason is $(q_C, q_L) = (1/3, \pm 1/2)$ for the $(\mathbf{3}, \mathbf{2})$ representation, although its inclusion does not lead to values $k_Y < 5/3$ in any case.

Thus, we see that the minimal embedding of the MSSM gauge group and MSSM spectrum with the simplest moding $\Delta = 2$ leads naturally to the minimum hypercharge normalization $k_Y = 5/3$. This is, of course, the value that is obtained in field-theoretic GUT scenarios such as those making use of an $SO(10)$ embedding. The important point here, however, is that we have made absolutely no assumptions concerning any GUT scenario or embedding. Rather, we have employed only *string-based* arguments, and found that they too naturally lead to the same result.

As a final comment, we note that our restriction to charge vector components $Q_i \in \mathbb{Z}/2$ does *not* mean that the corresponding model cannot involve bosons with higher twists, or equivalently fermions with multiperiodic boundary conditions. This restriction merely implies that the MSSM chiral matter should not arise from the sectors with higher twists. In other words, higher-twist sectors may appear in the string model, but they should only have the effect of introducing extra non-MSSM matter and/or implementing higher-twist GSO projections on the MSSM sectors. This is the certainly the case with most of the realistic free-fermionic string models that have been examined in the literature, such as those of Refs. [13, 26, 15]. Even though these models contain higher twists, they do not have chiral matter coming from those sectors. Thus they are subject to our result as well, and since they exhibit hypercharges of the “minimal” form (4.12), they always have $k_Y \geq 5/3$.

5 Hypercharge Embeddings: Higher Twists and Higher Kač-Moody Levels

In this section we apply the methods developed in the previous section in order to consider two different extensions of those results. The first extension involves higher twists, with charge modings $\Delta > 2$, and we will see in this case (unlike the case for $\Delta = 2$), there *do* exist isolated hypercharge embeddings with $k_Y < 5/3$. The second extension involves the generalization to higher-level non-abelian gauge symmetries — in particular, to realizations of the non-abelian MSSM gauge group $SU(3) \times SU(2)$ at levels $k_2, k_3 > 1$. We shall find that this too can have strong effects on the possible hypercharge embeddings.

5.1 Higher Twists

We first consider the cases with higher values of the moding Δ . As we have seen, restricting ourselves to $\Delta = 2$ leads naturally to the conclusion that $k_Y \geq 5/3$. However, it is possible *a priori* that the MSSM representations can arise from sectors with higher twists, *e.g.*, sectors with $\Delta = 4$ modings. Thus such cases must be analyzed as well. As we shall see, in these cases the number of possible embeddings is much larger, and several isolated embeddings within this class do have $k_Y < 5/3$.

Whether these embeddings can be realized within consistent string *models* remains, however, an open question.

At first glance, one might suspect that chiral matter cannot possibly arise from sectors with $\Delta > 2$ modings; the (ultimately faulty) line of reasoning would be that the presence of such higher-twist sectors in a bosonic or fermionic construction corresponds to Wilson lines in an orbifold formulation, and Wilson lines can only introduce non-chiral matter. If this were true, our results from the last section would then be completely general, and we would have shown that $k_Y \geq 5/3$ in all level-one free-fermionic (or more generally, in all level-one free-field) string models employing minimal hypercharge embeddings of the form (4.12). Unfortunately, however, the correspondence between such higher-twist sectors and Wilson lines is not as precise as would be required for this type of claim, and furthermore the assertion that Wilson lines do not give chiral matter relies upon the compactification manifold being sufficiently smooth.* In particular, if there exist singularities (such as exist in orbifold compactifications, for example), then Wilson lines may give additional chiral matter. Consequently, it is not possible to rule out the appearance of chiral matter originating from higher-twist sectors. Indeed, a simple example of a model in which chiral matter *does* arise in such higher-twist sectors (in this case, sectors with $\Delta = 6$) can be found in Ref. [27]. It is therefore also necessary to examine the cases with higher values of Δ , and in this subsection we shall analyze the next-simplest case with $\Delta = 4$.

Analyzing the set of possible hypercharge embeddings for $\Delta = 4$ can be done in the same manner as for $\Delta = 2$. Imposing the masslessness, chirality, and moding restrictions, it turns out that there are 16 possible combinations of (q_C, q_L) charges for the $(\mathbf{3}, \mathbf{2})$ representation of $SU(3) \times SU(2)$, 25 possible combinations for the $(\bar{\mathbf{3}}, \mathbf{1})$ representation, 30 combinations for the $(\mathbf{1}, \mathbf{2})$ representation, and 36 for the $(\mathbf{1}, \mathbf{1})$. Including the Higgs H^+ and H^- representations, this yields potentially $16 \cdot (25)^2 \cdot (30)^3 \cdot 36 = 9.72 \times 10^9$ different sets of charge vectors. Of this number, however, only a relative few are consistent with a hypercharge assignment of the general form (4.12), and indeed the overwhelming majority of those have corresponding values of k_Y greater than or equal to $5/3$.

Remarkably, however, there are six isolated embeddings which yield consistent hypercharge assignments with $k_Y < 5/3$. In order of increasing k_Y , the values obtained are

$$k_Y = \frac{53}{48}, \frac{100}{81}, \frac{212}{147}, \frac{116}{75}, \frac{77}{48}, \frac{236}{147}, \quad (5.1)$$

which stretch within the range $1.10 \leq k_Y \leq 1.61$.

Cubic-Level Mass Terms

In order to further restrict this set, it is possible to impose one additional constraint. As we have discussed in the previous section, the MSSM hypercharge assign-

* We thank I. Antoniadis and E. Witten for discussions on this point.

ments are chosen in part so that the mass terms listed in (A.3) will be invariant under $U(1)_Y$. This implies, of course, that the sum of the hypercharges of the three representations within each mass term must be zero. Within a *string-theoretic* framework, however, such cubic mass terms (or more generally, such direct couplings between any number of fields) are allowed in the low-energy effective superpotential only if the sum of the entire corresponding charge *vectors* vanishes. Thus, not only must the hypercharge linear combination $\sum_{\text{reps}} \sum_i a_i Q_i$ vanish, but in fact the entire vector $\sum_{\text{reps}} \mathbf{Q}$ must vanish. This implies that the sums of the q_C and q_L quantum numbers must vanish separately.

While the cancellation of the hypercharge has been assured by our hypercharge assignments from the beginning, the cancellation of the entire charge vectors is a completely separate matter. For example, it is possible to realize an *effective* mass term between three fields Φ_i , Φ_j , and Φ_k even if the sum of their charge vectors $\mathbf{Q}_i + \mathbf{Q}_j + \mathbf{Q}_k$ does not cancel: all that is required is a fourth field Φ_ℓ whose charge vector exactly cancels the sum, so that the quartic coupling $\Phi_i \Phi_j \Phi_k \Phi_\ell$ is allowed. Giving a VEV to Φ_ℓ then introduces the effective mass term

$$\langle \Phi_\ell \rangle \Phi_i \Phi_j \Phi_k \quad (5.2)$$

whose strength is set by the value of the VEV. Such a mass term is said to arise at the quartic level in the superpotential, and higher-order terms are also possible. Thus, strictly speaking, it is not required for the existence of an *effective* mass term that the sum of the charge vectors cancel; indeed, this requirement holds only for mass terms that arise at the cubic level in the superpotential.

There are, however, both stringy and phenomenological reasons for distinguishing between the cubic mass terms and the higher-order mass terms. On the string side, if we do not require that the sum of the charge vectors cancel, then we must instead demand that there simultaneously exist in the spectrum an additional field (or product of fields) whose total charge vector cancels $\mathbf{Q}_i + \mathbf{Q}_j + \mathbf{Q}_k$, and which is also invariant under the entire gauge symmetry $SU(3) \times SU(2) \times U(1)_Y$ (so that its acquisition of a VEV does not break the MSSM gauge symmetry). This is a difficult constraint to implement in a model-independent fashion, and thus it is natural to demand (for simplicity) that *all* mass terms arise at only the cubic level of the superpotential. Likewise, on the phenomenological side, it turns out that the order of such a mass term typically dictates the *scale* of the corresponding coupling. While the scale of the cubic mass terms is typically fixed by $SL(2, \mathbb{C})$ invariance, the higher-order mass terms generally appear with inverse factors of the Planck mass, and therefore yield effective mass terms that are increasingly suppressed. It is therefore reasonable to impose that at least one mass term be invariant at the cubic level. Such a mass term will then be identified with the top quark mass term, and constrain the allowed embeddings. In a similar way, we might additionally demand, for example, that *all* of the heavy fermion mass terms arise at the cubic level, and so forth. Thus, for both stringy and phenomenological reasons, it is natural to impose the additional

constraint on our embeddings that they are consistent with each of the mass terms in (A.3) arising at the cubic level of the superpotential.

If we impose this additional constraint, we then find that only two of our six embeddings with $k_Y < 5/3$ survive. These are the embeddings with $k_Y = 77/48 \approx 1.604$ and $k_Y = 236/147 \approx 1.605$, and are as follows. The $k_Y = 77/48$ embedding consists of the charges

$$\begin{array}{ll}
Q : & (q_C, q_L) = (-1/6, 1/2) \quad \mathbf{Q} = (-1/2, -1/2, 1/2, 0, 1) \\
u : & (q_C, q_L) = (-1/12, -3/4) \quad \mathbf{Q} = (1/4, 1/4, -3/4, -3/4, -3/4) \\
d : & (q_C, q_L) = (5/12, -1/4) \quad \mathbf{Q} = (3/4, 3/4, -1/4, -1/4, -1/4) \\
L : & (q_C, q_L) = (-1/4, -1/4) \quad \mathbf{Q} = (-1/4, -1/4, -1/4, -3/4, 1/4) \\
e : & (q_C, q_L) = (1/2, 1/2) \quad \mathbf{Q} = (1/2, 1/2, 1/2, 1/2, 1/2) \\
H^+ : & (q_C, q_L) = (1/4, 1/4) \quad \mathbf{Q} = (1/4, 1/4, 1/4, 3/4, -1/4) \\
H^- : & (q_C, q_L) = (-1/4, -1/4) \quad \mathbf{Q} = (-1/4, -1/4, -1/4, -3/4, 1/4) , \quad (5.3)
\end{array}$$

and corresponds to the hypercharge solution $(A_1, A_2) = (5/12, 3/8)$. Likewise, the $k_Y = 236/147$ embedding consists of the charges

$$\begin{array}{ll}
Q : & (q_C, q_L) = (1/3, -1/4) \quad \mathbf{Q} = (0, 0, 1, -1/4, 3/4) \\
u : & (q_C, q_L) = (-7/12, 0) \quad \mathbf{Q} = (-1/4, -1/4, -5/4, 0, 0) \\
d : & (q_C, q_L) = (-1/12, -1/2) \quad \mathbf{Q} = (1/4, 1/4, -3/4, -1/2, -1/2) \\
L : & (q_C, q_L) = (-1/4, 1/4) \quad \mathbf{Q} = (-1/4, -1/4, -1/4, -1/4, 3/4) \\
e : & (q_C, q_L) = (1/2, -1/2) \quad \mathbf{Q} = (1/2, 1/2, 1/2, -1/2, -1/2) \\
H^+ : & (q_C, q_L) = (1/4, -1/4) \quad \mathbf{Q} = (1/4, 1/4, 1/4, 1/4, -3/4) \\
H^- : & (q_C, q_L) = (-1/4, 1/4) \quad \mathbf{Q} = (-1/4, -1/4, -1/4, -1/4, 3/4) , \quad (5.4)
\end{array}$$

and corresponds to the hypercharge solution $(A_1, A_2) = (8/21, -3/7)$. We find it remarkable that using only quarter-integer components, there exist embeddings which have the required forms given in (4.6) and (4.7), which are simultaneously consistent with all of our constraints (*e.g.*, masslessness, chirality, and cubic-level mass terms), and which also yield the correct MSSM hypercharge assignments via linear combinations corresponding to normalizations $k_Y < 5/3$.

Existence of $k_Y < 5/3$ String Models?

Given the unique nature of these embeddings with $k_Y < 5/3$, the next step is to attempt to realize them in a consistent string model. After all, the embeddings that we have constructed are meaningful only if they can be realized as the results of actual string models. This means that we must seek to construct a string model, complete with corresponding GSO projections, in such a way that all of the following conditions are met. The first class of conditions, of course, are the model-independent

constraints which guarantee that our model is self-consistent: these include worldsheet conformal anomaly cancellation, modular invariance, proper worldsheet supercurrent, proper spin-statistics, *etc.* Next, our second class of constraints are those that pertain to the particular models we seek to construct. In particular, we must choose our GSO constraints in such a way that:

- the model has $N = 1$ spacetime supersymmetry;
- the appropriate gauge group $SU(3) \times SU(2) \times U(1)_Y$ is realized;
- each of the desired MSSM representations in (A.1) appears (*i.e.*, survives the GSO constraints);
- each such representation appears *chirally* (*i.e.*, the chiral conjugates of each MSSM representation must *not* appear, and must therefore be eliminated by the GSO constraints); and
- the MSSM representations that survive the GSO constraints each have the same helicity.

Third, there are various additional constraints that can be verified only after a particular model satisfying the above constraints is constructed. For example, we must impose $U(1)_Y$ anomaly cancellation. Although we are guaranteed by our original hypercharge assignments that the MSSM representations by themselves are anomaly-free, one must nevertheless verify that all of the *additional* states that necessarily appear in the full string spectrum are also anomaly-free with respect to $U(1)_Y$. Given the absence of the standard $SO(10)$ embeddings or gauge symmetry, such anomaly-cancellations are not at all guaranteed. Finally, additional constraints that we might choose to impose for “realistic” models might be constraints on the numbers of generations, and so forth. Potential constraints arising from the appearance (or avoidance) of fractionally-charged states will be discussed in the next section.

To date, we have not succeeded in building a self-consistent string model which realizes these “minimal” $k_Y < 5/3$ embeddings and which simultaneously satisfies the first and second groups of constraints. The primary difficulty arises in realizing a set of GSO projections which is self-consistent (*i.e.*, which does not violate the first group of constraints), but which yields the necessary chirality properties for the MSSM representations as listed above. Thus, the existence of consistent string models realizing these minimal $k_Y < 5/3$ embeddings remains an open question.

5.2 Higher Kač-Moody Levels

We now consider the possible hypercharge embeddings that can be constructed when the $SU(3)$ and $SU(2)$ gauge factors are realized at Kač-Moody levels $k_3, k_2 > 1$. There are many reasons why it is important to examine such situations. For example, although the level-one models are the easiest to construct within certain string

formulations, the higher-level models are also within the general moduli space of self-consistent models, and moreover these models have certain appealing phenomenological features (such as gauge group rank-reduction, and potentially smaller sets of massless states and moduli). It is therefore of interest, for the purposes of gauge coupling unification, to examine what values of k_Y are possible for such models. Second, a more practical reason for examining the higher-level cases is that our analysis up to this point is highly dependent on the fact that we have been working at level $k_2 = k_3 = 1$. Indeed, this choice directly determined the minimal gauge-group embedding, which in turn dictated the possible matter embeddings, and likewise the possible hypercharge embeddings. Higher levels for the non-abelian group factors should therefore profoundly change the spectrum of possible values of k_Y . Third, as we have seen in Sect. 3, for the purposes of gauge coupling unification, it is not really k_Y which is of interest, but rather the quotient k_Y/k_2 . Hence, even if (for some reason) one cannot realize $k_Y < 5/3$ in a realistic level-one models, it may still be possible to realize $k_Y < 10/3$ in a level-*two* model. Finally, yet another reason for examining the cases with higher levels has to do with charge integrality. As we shall see in the next section, the values of k_Y that one can obtain in string models containing only integrally charged color-neutral states are tightly constrained by the values of k_2 and k_3 . This deep relation between k_Y and the levels of the non-abelian factors therefore suggests that a constructive examination of possible hypercharge embeddings for higher-level models is in order.

In this section, we shall examine the case with $k_2 = k_3 = 2$. As we have seen in Sect. 3 [particularly in Fig. 4 and Eq. (3.19)], models with $k_2 = k_3 = 2$ are still within the phenomenologically preferred ranges for gauge coupling unification. Furthermore, we shall find that the $k_2 = k_3 = 2$ case is also relatively simple to analyze in the manner outlined in the previous section, and we shall discuss the situations for charge modings $\Delta = 2$ and $\Delta = 4$. Extensions to higher levels and/or modings can then be handled in a similar fashion.

Gauge Group Embeddings

We begin by considering the possible embeddings of the gauge groups $SU(3)$ and $SU(2)$ realized at levels $k_3 = k_2 = 2$. From (4.1), we see that whereas we previously required roots of $(\text{length})^2 = 2$, we now require roots of $(\text{length})^2 = 1$. Given this constraint, we then find that at least *four* lattice components are necessary for embedding the root system of $SU(3)_2$, with the two simple roots of $SU(3)$ chosen to have the coordinates

$$(1, 0, 0, 0) \quad \text{and} \quad (-1/2, 1/2, -1/2, 1/2) . \quad (5.5)$$

As required, these two roots have a relative angle of 120° . The fact that $SU(3)$ has rank two implies that there are now *two* directions orthogonal to the $SU(3)$ hyperplane, and in the coordinate system set by (5.5) these two directions can be

taken to be

$$Q_A \equiv (0, 1, 1, 0) \quad \text{and} \quad Q_B \equiv (0, 0, 1, 1) . \quad (5.6)$$

For $SU(2)$, on the other hand, the minimal embedding is particularly simple at level $k_2 = 2$, and requires only *one* lattice direction. We thus assign the single simple root of $SU(2)$ to have lattice coordinate (1). The fact that such a one-dimensional embedding is possible implies that there is no additional “orthogonal” component which must be introduced into this minimal embedding.

We see, then, that just as in the level-one case, the minimal embedding of the root system of $SU(3)_2 \times SU(2)_2$ requires a *five*-dimensional charge lattice, and we shall once again denote these lattice components as Q_i , $i = 1, \dots, 5$, where now $i = 1, \dots, 4$ denote the $SU(3)_2$ factor (and its two orthogonal directions), and $i = 5$ denotes the $SU(2)_2$ factor. Despite this superficial similarity to the level-one case, however, there are some strong distinctions. The most important of these concerns the appearance of *rank-cutting* and of extra non-gauge chiral algebras. As we can see from (2.4), the central charges of these gauge group factors are respectively $c_{SU(3)} = 16/5$ and $c_{SU(2)} = 3/2$ at level two. This means that although we are assigning two lattice directions to $SU(3)$ and one lattice direction to $SU(2)$, in reality the full $SU(3)$ conformal field theory must involve some additional $c = 6/5$ worth of worldsheet degrees of freedom which are disconnected from the two lattice directions we are considering here; likewise the full $SU(2)$ conformal field theory requires an additional $c = 1/2$ worth of worldsheet degrees of freedom which are not reflected in the gauge charge lattice. This means that unlike the level-one case, certain worldsheet degrees of freedom do not contribute to the charge lattice of the model. Indeed, it is in this manner that the rank-cutting occurs, whereby the dimension of the total charge lattice of the model is reduced. In such theories, therefore, we see that the worldsheet excitations that produce, *e.g.*, the non-Cartan gauge bosons of higher-level gauge symmetries have two components: some of the excitations are among those worldsheet fields (*e.g.*, bosons or complex fermions) which contribute to the charge lattice (thereby creating a non-zero charge vector), and other excitations are among those worldsheet fields (such as the *necessarily real* fermions) that do not have a corresponding charge lattice. It is for this reason that the charge vectors can be reduced in length without altering the conformal dimensions of the corresponding gauge boson states. The same observations apply to the matter representations as well.

Embeddings for Matter Representations

Given the above embeddings for the root systems of $SU(3)_2$ and $SU(2)_2$, it is then straightforward to deduce the corresponding embeddings for the relevant matter

representations. We find

$$\begin{aligned}
(\mathbf{3})_{(q_A, q_B)} \text{ of } SU(3) : & \quad \left\{ \begin{array}{l} (0, 1/3, -1/3, 1/3) \\ (1/2, -1/6, 1/6, -1/6) \\ (-1/2, -1/6, 1/6, -1/6) \end{array} \right\} \\
& \quad + q_A (0, 1, 1, 0) + q_B (0, 0, 1, 1) \\
(\mathbf{1})_{(q_A, q_B)} \text{ of } SU(3) : & \quad q_A (0, 1, 1, 0) + q_B (0, 0, 1, 1) \\
(\mathbf{2}) \text{ of } SU(2) : & \quad \pm(1/2) \\
(\mathbf{1}) \text{ of } SU(2) : & \quad (0) .
\end{aligned} \tag{5.7}$$

As in the level-one case, we again have two quantum numbers q_A and q_B whose values are *a priori* arbitrary. Unlike the level-one case, however, these two quantum numbers are now both attached to the $SU(3)$ representations, and the $SU(2)$ representations have no remaining degrees of freedom.

Conformal Dimensions

We now consider the conformal dimensions of these representations (in order to eventually enforce our masslessness conditions). Unlike the level-one case, however, the conformal dimension is no longer given by (4.8); this occurs because in models with rank-cutting, the conformal dimensions receive contributions not only from the charge lattice excitations [which are tallied in (4.8)], but also from the extra world-sheet degrees of freedom which do not contribute to the charge lattice [and which are therefore not reflected in (4.8)]. It is necessary, therefore, to use the more general expressions (2.5) when computing the conformal dimensions of each given representation. In this manner we then determine the conformal dimensions for the relevant representations of $SU(3)_2 \times SU(2)_2$:

$$\begin{aligned}
(\mathbf{3}, \mathbf{2})_{(q_A, q_B)} : & \quad h = 109/240 + q_A^2 + q_B^2 \\
(\overline{\mathbf{3}}, \mathbf{1})_{(q_A, q_B)} : & \quad h = 4/15 + q_A^2 + q_B^2 \\
(\mathbf{1}, \mathbf{2})_{(q_A, q_B)} : & \quad h = 3/16 + q_A^2 + q_B^2 \\
(\mathbf{1}, \mathbf{1})_{(q_A, q_B)} : & \quad h = q_A^2 + q_B^2 .
\end{aligned} \tag{5.8}$$

Of course, the allowed values of (q_A, q_B) for each representation are then constrained by the requirement that the corresponding conformal dimension must be less than one, and likewise by whatever moding requirements we wish to impose. We also impose the chirality condition discussed in the previous section. In any case, however, the ultimate constraint on values of (q_A, q_B) for each representation comes from the requirement that a consistent hypercharge assignment for each of the MSSM representations must be simultaneously realizable.

Hypercharge Embeddings

We now approach the question of the corresponding hypercharge embeddings. As before, we consider only the “minimal” embedding of the form (4.12), and determine the solutions for which the corresponding values of k_Y are in the appropriate range.

Given our hypercharge embedding of the form (4.12), we immediately observe that for the higher-level matter embeddings in (5.7), we now must have

$$a_1 = a_5 = 0 \quad \text{and} \quad a_2 + a_4 = a_3 . \quad (5.9)$$

These constraints are necessary in order to ensure that the same hypercharge value Y is obtained for each charge vector within a single representation. Thus, we see that once again there are only two independent coefficients, a_2 and a_4 , which are unconstrained. As expected, this renders the hypercharge assignment independent of the particular $SU(3)$ or $SU(2)$ representations \mathbf{R} or \mathbf{R}' , so that Y depends on only the (q_A, q_B) quantum numbers:

$$Y(\mathbf{R}_{(q_A, q_B)}, \mathbf{R}') = a_2 q_A + (a_2 + a_4)(q_A + q_B) + a_4 q_B . \quad (5.10)$$

For any such solution (a_2, a_4) , the corresponding value of k_Y is then given by

$$k_Y = 2 \sum a_i^2 = 2 \left[a_2^2 + (a_2 + a_4)^2 + a_4^2 \right] . \quad (5.11)$$

The task therefore remains to choose a particular value of moding Δ , and to determine the possible corresponding embeddings for which a hypercharge may be consistently defined.

We first present results for the moding $\Delta = 2$. Choosing this moding amounts to the restriction that q_A and q_B be in the set $1/6 + \mathbb{Z}/2$ for the triplet $\mathbf{3}$ representation of $SU(3)$, or in the set $\mathbb{Z}/2$ for the singlet representation of $SU(3)$. Imposing the masslessness and chirality constraints then yields a finite set of possibilities of (q_A, q_B) for each representation. This ultimately yields several hundred thousand combined possibilities for the charges (q_A, q_B) of the representations (Q, u, d, L, e, H^+) . However, we find that only for a relatively small set are consistent hypercharge assignments possible, and indeed the values of k_Y to which they correspond are:

$$k_Y = \frac{4}{3}, \frac{13}{3}, \frac{28}{3}, \frac{52}{3} . \quad (5.12)$$

The embedding with $k_Y = 52/3$ is in fact unique (*i.e.*, there is only one embedding with this value), and it does not satisfy the cubic-level mass-term constraints discussed in the previous section. By contrast, among the embeddings corresponding to each other value of k_Y , there are some which satisfy these cubic-level mass-term constraints, and some which do not.

It is clear from these results, then, that the spectrum of k_Y values realizable in such embeddings is quite limited; moreover, the value $k_Y = 4/3$ is special in that is already less than $5/3$ (with the value of k_Y/k_2 less than one, no less!). It turns out

that there are two independent hypercharge embeddings which take this value of k_Y and which also satisfy all of the cubic-level mass-term constraints. These embeddings are respectively

$$\begin{aligned}
Q : & \quad (q_A, q_B) = (-1/3, 1/6) & \mathbf{Q} &= (0, 0, -1/2, 1/2, 1/2) \\
u : & \quad (q_A, q_B) = (1/3, 1/3) & \mathbf{Q} &= (0, 0, 1, 0, 0) \\
d : & \quad (q_A, q_B) = (1/3, -2/3) & \mathbf{Q} &= (0, 0, 0, -1, 0) \\
L : & \quad (q_A, q_B) = (1/2, 0) & \mathbf{Q} &= (0, 1/2, 1/2, 0, 1/2) \\
e : & \quad (q_A, q_B) = (-1/2, -1/2) & \mathbf{Q} &= (0, -1/2, -1, -1/2, 0) \\
H^+ : & \quad (q_A, q_B) = (0, -1/2) & \mathbf{Q} &= (0, 0, -1/2, -1/2, 1/2) \\
H^- : & \quad (q_A, q_B) = (0, 1/2) & \mathbf{Q} &= (0, 0, 1/2, 1/2, -1/2) , \quad (5.13)
\end{aligned}$$

and

$$\begin{aligned}
Q : & \quad (q_A, q_B) = (-1/3, 1/6) & \mathbf{Q} &= (0, 0, -1/2, 1/2, 1/2) \\
u : & \quad (q_A, q_B) = (5/6, -1/6) & \mathbf{Q} &= (0, 1/2, 1, -1/2, 0) \\
d : & \quad (q_A, q_B) = (-1/6, -1/6) & \mathbf{Q} &= (0, -1/2, 0, -1/2, 0) \\
L : & \quad (q_A, q_B) = (0, 1/2) & \mathbf{Q} &= (0, 0, 1/2, 1/2, 1/2) \\
e : & \quad (q_A, q_B) = (-1/2, -1/2) & \mathbf{Q} &= (0, -1/2, -1, -1/2, 0) \\
H^+ : & \quad (q_A, q_B) = (-1/2, 0) & \mathbf{Q} &= (0, -1/2, -1/2, 0, 1/2) \\
H^- : & \quad (q_A, q_B) = (1/2, 0) & \mathbf{Q} &= (0, 1/2, 1/2, 0, -1/2) , \quad (5.14)
\end{aligned}$$

both of which correspond to the solution $a_2 = a_4 = -1/3$. The fact that there are no embeddings with values of k_Y in the range $5/3 \leq k_Y < 10/3$ is unfortunate, however, as we *a priori* seek values $k_Y \approx 2.8$ for these $k_2 = 2$ embeddings.

This deficiency can be overcome, however, by considering the more general moding $\Delta = 4$, and indeed this introduces additional potential hypercharge embeddings which satisfy all of the cubic-level mass-term constraints and which do have $k_Y/k_2 < 5/3$. In fact, the particular set of values of k_Y to which such embeddings correspond is remarkably limited. We find, for example, that for $\Delta = 4$, there exist such potential embeddings with only the following values of $k_Y < 5/3$:

$$k_Y = \frac{4}{3}, \frac{112}{81}, \frac{208}{147}, \frac{112}{75} . \quad (5.15)$$

Similarly, within the range $5/3 \leq k_Y < 10/3$, the only values of k_Y for which such embeddings exist are the following:

$$k_Y = \frac{16}{9}, \frac{7}{3}, \frac{208}{75}, \frac{28}{9} . \quad (5.16)$$

We see from these results, then, that there exist embeddings with $k_Y \approx 2.8$ which are potentially realizable in $k_2 = 2$ string models, and which satisfy all of our

constraints (including all of the cubic-level mass-term constraints). In particular, these are the embeddings with $k_Y = 208/75 \approx 2.8$, so that $k_Y/k_2 \approx 1.4$. It turns out that there are three distinct embeddings with this value of k_Y (modulo trivial sign flips and permutations). The first corresponds to the hypercharge solution $(a_2, a_4) = (4/15, 2/3)$, and is

$$\begin{aligned}
Q : & \quad (q_A, q_B) = (-1/12, 1/6) & \mathbf{Q} &= (0, 1/4, -1/4, 1/2, 1/2) \\
u : & \quad (q_A, q_B) = (1/3, -2/3) & \mathbf{Q} &= (0, 0, 0, -1, 0) \\
d : & \quad (q_A, q_B) = (-1/6, 1/3) & \mathbf{Q} &= (0, -1/2, 1/2, 0, 0) \\
L : & \quad (q_A, q_B) = (-3/4, 1/4) & \mathbf{Q} &= (0, -3/4, -1/2, 1/4, 1/2) \\
e : & \quad (q_A, q_B) = (1/2, 1/4) & \mathbf{Q} &= (0, 1/2, 3/4, 1/4, 0) \\
H^+ : & \quad (q_A, q_B) = (-1/4, 1/2) & \mathbf{Q} &= (0, -1/4, 1/4, 1/2, 1/2) \\
H^- : & \quad (q_A, q_B) = (1/4, -1/2) & \mathbf{Q} &= (0, 1/4, -1/4, -1/2, -1/2) . \quad (5.17)
\end{aligned}$$

The remaining two $k_Y = 208/75$ embeddings, by contrast, correspond to the different hypercharge solution $(a_2, a_4) = (-14/15, 2/3)$, and are

$$\begin{aligned}
Q : & \quad (q_A, q_B) = (-1/12, 1/6) & \mathbf{Q} &= (0, 1/4, -1/4, 1/2, 1/2) \\
u : & \quad (q_A, q_B) = (1/3, -2/3) & \mathbf{Q} &= (0, 0, 0, -1, 0) \\
d : & \quad (q_A, q_B) = (-1/6, 1/3) & \mathbf{Q} &= (0, -1/2, 1/2, 0, 0) \\
L : & \quad (q_A, q_B) = (1/2, 1/4) & \mathbf{Q} &= (0, 1/2, 3/4, 1/4, 1/2) \\
e : & \quad (q_A, q_B) = (-3/4, 1/4) & \mathbf{Q} &= (0, -3/4, -1/2, 1/4, 0) \\
H^+ : & \quad (q_A, q_B) = (-1/4, 1/2) & \mathbf{Q} &= (0, -1/4, 1/4, 1/2, 1/2) \\
H^- : & \quad (q_A, q_B) = (1/4, -1/2) & \mathbf{Q} &= (0, 1/4, -1/4, -1/2, -1/2) \quad (5.18)
\end{aligned}$$

and

$$\begin{aligned}
Q : & \quad (q_A, q_B) = (-1/12, 1/6) & \mathbf{Q} &= (0, 1/4, -1/4, 1/2, 1/2) \\
u : & \quad (q_A, q_B) = (7/12, 1/12) & \mathbf{Q} &= (0, 1/4, 1, -1/4, 0) \\
d : & \quad (q_A, q_B) = (-5/12, -5/12) & \mathbf{Q} &= (0, -3/4, -1/2, -3/4, 0) \\
L : & \quad (q_A, q_B) = (1/4, -1/2) & \mathbf{Q} &= (0, 1/4, -1/4, -1/2, 1/2) \\
e : & \quad (q_A, q_B) = (-3/4, 1/4) & \mathbf{Q} &= (0, -3/4, -1/2, 1/4, 0) \\
H^+ : & \quad (q_A, q_B) = (-1/2, -1/4) & \mathbf{Q} &= (0, -1/2, -3/4, -1/4, 1/2) \\
H^- : & \quad (q_A, q_B) = (1/2, 1/4) & \mathbf{Q} &= (0, 1/2, 3/4, 1/4, -1/2) . \quad (5.19)
\end{aligned}$$

It is clear, then, that within the context of higher-level realizations of the $SU(2)$ and $SU(3)$ gauge groups, there exist potential hypercharge embeddings with $k_Y/k_2 \approx 1.4$. Furthermore, the freedom to build embeddings beyond those of the form (4.12) should also enable other nearby values of k_Y to be reached, although we would *a priori* expect the inclusion of additional lattice components in

the hypercharge linear combination to tend to increase the corresponding values of k_Y . Nevertheless, the spectrum of allowed values of k_Y within just the “minimal” embeddings of the form (4.12) is sufficiently diverse to suggest that within the class of higher-twisted or higher-level string models, various phenomenologically interesting values of k_Y can be obtained without difficulty. Of course, it remains an open question as to whether these special embeddings can be realized within actual self-consistent string models. The existence of these special embeddings should nevertheless provide a useful starting point in this endeavor.

6 Charge Quantization Constraints

In the previous two sections, we analyzed the constraints on the values of k_Y that arise for various “minimal embeddings” of the MSSM gauge group. In this way we were able to obtain certain interesting results. For example, by proving the non-existence of $k_Y < 5/3$ embeddings for $\Delta = 2$ charge modings and level-one $SU(2)$ and $SU(3)$ gauge factors, we were able to prove that no such models utilizing such minimal embeddings can have $k_Y < 5/3$. Likewise, in the cases of more general modings and higher Kač-Moody levels, we were able to construct potentially viable embeddings that do have $k_Y < 5/3$ (or more generally $k_Y/k_2 < 5/3$). These embeddings should therefore prove useful in the construction of phenomenologically interesting $k_Y < 5/3$ string models.

As we discussed in Sect. 4.1, however, such an approach has the disadvantage that it is beyond the reach of certain powerful string consistency constraints such as modular invariance. In this section, therefore, we shall now follow a somewhat orthogonal approach to constraining the MSSM levels (k_Y, k_2, k_3) , one which incorporates modular invariance at an early stage. As we shall find, this method will allow us to correlate the combinations of Kač-Moody levels with the appearance of fractionally charged states in the string spectrum, and thereby constrain the phenomenologically preferred combinations of levels (k_Y, k_2, k_3) for which such states do not appear.

6.1 The Method

As originally shown by Schellekens [23], an interesting set of restrictions on the Kač-Moody levels (k_Y, k_2, k_3) arises by imposing charge-quantization conditions on the asymptotic states of the theory in the infrared. These conditions are motivated by the observation that in the observed MSSM particle spectrum, there is a strong correlation between the allowed combinations of $SU(3) \times SU(2) \times U(1)$ representations: all color singlets have integer electric charge (in units of the electron charge). Indeed, there are also quite stringent experimental constraints on the existence of massive fractionally charged states, stemming from the fact that despite various experimental efforts, fractionally charged states have not been seen [28]. In this section, therefore, we will consider the consequences of imposing a charge integrality condition on the

allowed states in string theory. First, as we shall see, if we simply demand that the spectrum of a given string model does not contain fractionally charged states *at all*, then it is straightforward to show that we must have in fact have $k_Y \geq 5/3$ for level-one models (*i.e.*, models with $k_2 = k_3 = 1$). This result is essentially contained within Schellekens' original analysis. However, string models generically *will* contain fractionally charged states; such states are then presumed to be *bound* into color singlets under the influence of some extra hidden confining gauge symmetries (or “hypercolor” groups) beyond those of the MSSM. We shall therefore generalize the Schellekens analysis for the cases of arbitrary hypercolor groups beyond that of color $SU(3)$, and require only the weaker constraint that all of the string states which appear can be bound into color singlets under the hypercolor interactions. This will then enable us to determine which more general combinations of Kač-Moody levels (and in particular which more general values of k_Y) are allowed. In a similar analysis, we shall also consider a slightly different generalization of the charge integrality constraints in order to determine the values of k_Y that can arise when only a restricted set of fractional charges are permitted to appear in the spectrum. Note that such charge-integrality constraints on the Kač-Moody levels have also been previously considered for the purposes of analyzing certain specific string models [29, 30].

The starting point of our analysis is a theorem proved by Schellekens [23] utilizing the properties of simple (or “bonus-symmetry”) currents that arise in many rational conformal field theories (RCFT's) [31, 32]. Recall that a rational conformal field theory is one that contains only finitely many primary fields. In such a theory, a simple current J is then defined as a primary field that has a one-term fusion rule with all primary fields ϕ_p in the theory, so that we obtain a fusion rule of the simple form

$$J \times \phi_p = \phi_{p'} . \quad (6.1)$$

Note that for each primary field ϕ_p , the resulting field $\phi_{p'}$ is unique. Thus, when acting on primary fields ϕ_p , each simple current J generates a cyclic action of order N , where N is the smallest integer such that $J^N = \mathbf{1}$ under fusion. In the following, we shall denote the identity current as $J_0 \equiv \mathbf{1}$. Note that the order N must be finite due to our restriction to RCFT's. Also note that, by construction, the identity operator J_0 is always a simple current of order $N = 1$. With these definitions, we then have:

Theorem.[23] For any rational unitary conformal field theory, consider the partition function

$$Z(\tau, \bar{\tau}) = \sum_{m, \tilde{n}} \chi_m(\tau) M_{m\tilde{n}} \tilde{\chi}_{\tilde{n}}(\bar{\tau}) , \quad (6.2)$$

where $M_{m\tilde{n}}$ is a matrix of positive integers, and where the left and right-moving sets of characters χ and $\tilde{\chi}$ can be different. Denote the primary fields corresponding to a left-right combination of characters as $\phi_{m\tilde{n}}$. Suppose that p and q are labels of simple

currents, such that $\phi_{p\bar{q}}$ is local with respect to all other primary fields $\phi_{m\bar{n}}$. Let us also suppose that $M_{m\bar{n}} \neq 0$, so that the primary fields $\phi_{m\bar{n}}$ are present in the theory. Then the modular invariance of the partition function $Z(\tau, \bar{\tau})$ requires that $M_{p\bar{q}} \neq 0$. Hence the primary field $\phi_{p\bar{q}}$ must also be present in the theory.

In this theorem, the condition of locality is the statement that in operator products of $J(z)$ with any primary field $\phi(w)$, we have

$$J(z)\phi(w) \sim \frac{\phi'(w)}{(z-w)^\alpha} + \dots \quad (6.3)$$

where the exponent $\alpha \equiv h(J) + h(\phi) - h(\phi')$ is an integer. (Likewise, we have similar relations for the anti-holomorphic OPE's; these will not be written in what follows.) Note that the *monodromy* of the current J with respect to a given field ϕ is defined as

$$\text{mono}(J, \phi) = h(J) + h(\phi) - h(\phi') \pmod{1} \quad (6.4)$$

where ϕ' is the primary field obtained in (6.3). Thus, locality is equivalent to requiring that $\text{mono}(J, \phi) \equiv 0$ for all ϕ . Note also that

$$\text{mono}(J \times J', \phi) = \text{mono}(J, \phi) + \text{mono}(J', \phi). \quad (6.5)$$

We shall have occasion to use this result later.

An important point to notice is that the conditions stated in the above theorem are not sufficient to guarantee the existence of a non-trivial modular invariant partition function including $\phi_{p\bar{q}}$. Indeed, there is an extra condition that must be satisfied by $\phi_{p\bar{q}}$: its holomorphic and anti-holomorphic conformal dimensions $(h(p), \bar{h}(q))$ must differ by an integer in order for the partition function to be invariant under the modular transformation $T : \tau \rightarrow \tau + 1$. As we shall see, this consistency condition on the conformal dimension of the simple currents will lead to a condition on the levels of the Kač-Moody algebras, for the conformal dimensions of the simple currents depend on the levels k_i . Thus, the logic of Schellekens argument is as follows. First, we demonstrate that given the charge-quantization condition (which is to be satisfied by all of the string states), it is possible to choose a combination \tilde{J} of simple currents such that the monodromy of \tilde{J} with respect to these states is *automatically* integral. By the theorem quoted above, this implies that \tilde{J} must appear as a primary field in the theory. Second, we then demand that the holomorphic and anti-holomorphic conformal dimensions of \tilde{J} differ by integers. This will then enable us to obtain a relation among the allowed Kač-Moody levels k_i .

6.2 Simple Currents for Classical Lie Algebras

Towards this end, our first concern is with the simple currents that occur in Kač-Moody algebras. For the Kač-Moody algebras associated with the classical Lie

algebras (which are the only cases we will consider), the enumeration of the non-trivial simple currents at level k is straightforward [31, 32, 33, 34]. Of course, for each case, the identity $J_0 \equiv \mathbf{1}$ is always a simple current. We shall now list the remaining non-trivial currents that arise for each group.

$SU(n+1)_k$: For $A_n \equiv SU(n+1)$ at level k , there are $n+1$ simple currents which we denote $J_A^{(m)}$ with $m = 0, \dots, n$. The case with $m = 0$ is the identity current J_0 , and the remaining currents $J_A^{(m)}$ with $m = 1, \dots, n$ are specified by the $SU(n+1)$ representations with Dynkin indices $(0, \dots, 0, k, 0, \dots, 0)$ where the non-zero entry is the m^{th} index. Note that under fusion, we have $(J_A^{(1)})^{n+1} = J_0$. The conformal dimensions of these currents are

$$h(J_A^{(m)}) = \frac{k m (n+1-m)}{2(n+1)}, \quad (6.6)$$

in accordance with the general expression (2.5). The other information that we will need is the monodromy of these currents with any primary field ϕ transforming in an arbitrary irreducible representation R of $SU(n+1)_k$. This monodromy is given by

$$\text{mono}(J_A^{(m)}, R) = \frac{m c_{su(n+1)}(R)}{n+1}, \quad (6.7)$$

where $c_{su(n+1)}(R)$ is the so-called congruence class [or “ $(n+1)$ -ality”] of the representation R in $SU(n+1)$. This congruence class is an integer modulo $(n+1)$, and is defined in terms of the Dynkin indices $a_{(j)}$ of R as

$$c_{su(n+1)}(R) \equiv \sum_{j=1, \dots, n} j a_{(j)}. \quad (6.8)$$

$SO(2n+1)_k$: For $B_n \equiv SO(2n+1)$ at level k , there is just one non-trivial simple current J_B . This current has Dynkin indices $(k, 0, \dots, 0)$, and its conformal dimension is

$$h(J_B) = k/2. \quad (6.9)$$

Likewise, its monodromy with respect to any representation R of $SO(2n+1)_k$ is

$$\text{mono}(J_B, R) = \frac{1}{2} c_{so(2n+1)}(R), \quad (6.10)$$

where the congruence class $c_{so(2n+1)}(R)$ is defined mod 2, and just labels whether R is vector-like (with $c = 0$) or spinor-like (with $c = 1$).

$Sp(2n)_k$: For $C_n \equiv Sp(2n)$ at level k , there is again just one non-trivial simple current J_C ; this current has Dynkin indices $(0, \dots, 0, k)$, and has conformal dimension

$$h(J_C) = \frac{nk}{4}. \quad (6.11)$$

Likewise, the monodromy of this current with respect to any representation R of $Sp(2n)_k$ is

$$\text{mono}(J_C, R) = \frac{1}{2} c_{sp(2n)}(R) , \quad (6.12)$$

where the congruence class $c_{sp(2n)}(R)$ is again defined mod 2, and now corresponds to the reality or pseudoreality of the representation R (with $c = 0$ for real representations, and $c = 1$ for pseudo-real representations).

$SO(2n)_k$: Finally, the $D_n \equiv SO(2n)$ algebras at level k possess three non-trivial simple currents, $J_D^{(v)} = (k, 0, \dots, 0)$, $J_D^{(s)} = (0, \dots, 0, k)$, and $J_D^{(c)} = (0, \dots, 0, k, 0)$, corresponding to the vector, spinor, and conjugate-spinor representations. These currents have conformal dimensions

$$h(J_D^{(v)}) = \frac{k}{2} , \quad h(J_D^{(s)}) = \frac{kn}{8} , \quad h(J_D^{(c)}) = \frac{kn}{8} , \quad (6.13)$$

and satisfy the fusion rules $J_D^{(v)} \times J_D^{(s)} = J_D^{(c)}$ and cyclic permutations. Unlike the previous cases, however, the structure of the monodromy of these currents with any given representation R of $SO(2n)_k$ depends on whether n is odd or even. This distinction ultimately arises because the center of $SO(2n)$ is \mathbb{Z}_4 if n is odd, but $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ if n is even. For n odd, we have

$$\begin{aligned} \text{mono}(J_v, R) &= \frac{1}{2} c_{so(4m+2)}(R) , \\ \text{mono}(J_s, R) &= \frac{1}{4} c_{so(4m+2)}(R) , \\ \text{mono}(J_c, R) &= \text{mono}(J_v, R) + \text{mono}(J_s, R) , \end{aligned} \quad (6.14)$$

where the congruence class $c_{so(4m+2)}(R)$ is defined (mod 4) in terms of the Dynkin indices of R by the explicit sum

$$c_{so(4m+2)}(R) = 2a_{(1)} + \dots + 2a_{(2m-3)} + (2m-1)a_{(2m)} + (2m+1)a_{(2m+1)} . \quad (6.15)$$

Note that although the congruence class is defined mod 4, all of the monodromies in (6.14) are nevertheless defined mod 1. Likewise, for n even, we have

$$\begin{aligned} \text{mono}(J_v, R) &= \frac{1}{2} c_{so(4m)}^{(1)}(R) , \\ \text{mono}(J_s, R) &= \frac{1}{2} c_{so(4m)}^{(2)}(R) , \\ \text{mono}(J_c, R) &= \text{mono}(J_v, R) + \text{mono}(J_s, R) , \end{aligned} \quad (6.16)$$

where now $c_{so(4m)}^{(i)}(R)$ ($i = 1, 2$) are the two components, both defined mod 2, of the charge conjugacy vector describing $\mathbb{Z}_2 \otimes \mathbb{Z}_2$. Explicitly, we have

$$\begin{aligned} c_{so(4m)}^{(1)}(R) &= a_{(2m-1)} + a_{(2m)} \\ c_{so(4m)}^{(2)}(R) &= a_{(1)} + a_{(3)} + \dots + a_{(2m-3)} + (m-1)a_{(2m-1)} + ma_{(2m)} . \end{aligned} \quad (6.17)$$

6.3 Charge Integrality Constraints with Confining Group

Given the above simple currents and monodromies, the first step in our derivation of the integrality constraint is to express the charge-quantization condition in terms of conjugacy classes. Let us generally take our low-energy gauge group to be of the form $G \times SU(3) \times SU(2) \times U(1)_Y$, where G is assumed to be semi-simple. Likewise, let us take each of the states ψ_i of our theory (whether massless or massive) to transform in some representation $(R_i, r_i, \tilde{r}_i, Y_i)$ of this group. The condition that all $SU(3)$ -color and G -hypercolor singlets be integrally charged with respect to the electromagnetic charge operator $Q = T_3 + Y$ then takes the form

$$\alpha_G(R_i) + \frac{c_{su(3)}(r_i)}{3} + \frac{c_{su(2)}(\tilde{r}_i)}{2} + Y_i \equiv 0 \pmod{1} \quad (6.18)$$

for each and every state ψ_i in the theory. Here the expression $\alpha_G(R)$ is an additive group-dependent quantity which is calculated as follows. For each gauge-group factor of $SU(n+1)$, $SO(2n+1)$, $Sp(2n)$, $SO(4m+2)$, or $SO(4m)$ which is present in G , $\alpha_G(R)$ is obtained by adding together the corresponding factors:

$$\begin{aligned} SU(n+1) : & \quad z c_{su(n+1)}(R)/(n+1) & \text{for } z = 0, \dots, n \\ SO(2n+1) : & \quad z c_{so(2n+1)}(R)/2 & \text{for } z = 0, 1 \\ Sp(2n) : & \quad z c_{sp(2n)}(R)/2 & \text{for } z = 0, 1 \\ SO(4m+2) : & \quad z c_{so(4m+2)}(R)/4 & \text{for } z = 0, 1, 2, 3 \\ SO(4m) : & \quad (z_1 c_{so(4m)}^{(1)}(R)/2 + z_2 c_{so(4m)}^{(2)}(R))/2 & \text{for } z_1, z_2 \in \{0, 1\} . \end{aligned} \quad (6.19)$$

The presence of the various integers z , z_1 , and z_2 in these formulas expresses the fact that, for a given confining group, there is more than one possible assignment of charges to the “quarks” that leads to only integrally charged bound states. In particular, if the chosen value of z divides the order of the center of the group (with common divisor d), then the minimally charged bound state has charge d in units of the electron charge. If, however, z does not divide the order, then the charges of the “hypercolor quarks” can be reassigned in a physically distinct way such that the minimal charge of the bound states remains one. These arbitrary factors of z are therefore inserted into (6.19) in order to reflect the freedom allowed by such reassignments of the electric charges.

Note that the fact that the condition (6.18) is necessary follows simply from the fact that singlets always live in the trivial conjugacy class. Likewise, the fact that this condition is sufficient follows from a consideration of the invariants of the classical Lie algebras. Also note that for the sake of notational brevity, we have dropped the species label i in (6.19) and in what follows.

The next step is to construct a combination \tilde{J} of the individual Kač-Moody simple currents in such a way that \tilde{J} is automatically local with respect to all states in the

theory by virtue of the quantization condition (6.18). For simplicity, let us suppose for the moment that G is simple. Then such a current-combination \tilde{J} which meets this condition is:

$$\tilde{J} = \left(J_G, J_{SU(3)}^{(1)}, J_{SU(2)}^{(1)}, \exp(i\sqrt{k_Y/2}\phi) \right). \quad (6.20)$$

Here ϕ is free boson that realizes $U(1)_Y$ via the current $J_Y \equiv i\sqrt{k_Y/2}\partial\phi$, normalized in accordance with the convention (2.11) for a $U(1)$ algebra at level k_Y . In (6.20), the choice for J_G depends on the choices for the particular factors (6.19) which enter into the expression $\alpha_G(R)$ in (6.18), as well as on the particular choices for the integers z (which are unconstrained). In particular, we see that these contributions to $\alpha_G(R)$ are successfully reproduced in the monodromy of \tilde{J} with a given state ψ_i if we choose

$$\begin{aligned} SU(n+1) : \quad J_G &= J_A^{(z)} \quad \text{for } z = 0, \dots, n \\ SO(2n+1) : \quad J_G &= \begin{cases} \mathbf{1} & \text{for } z = 0 \\ J_B & \text{for } z = 1 \end{cases} \\ SO(2n+1) : \quad J_G &= \begin{cases} \mathbf{1} & \text{for } z = 0 \\ J_C & \text{for } z = 1 \end{cases} \\ SO(4m+2) : \quad J_G &= \begin{cases} \mathbf{1} & \text{for } z = 0 \\ J_D^{(s)} & \text{for } z = 1 \\ J_D^{(v)} & \text{for } z = 2 \\ J_D^{(c)} & \text{for } z = 3 \end{cases} \\ SO(4m) : \quad J_G &= \begin{cases} \mathbf{1} & \text{for } (z_1, z_2) = (0, 0) \\ J_D^{(s)} & \text{for } (z_1, z_2) = (0, 1) \\ J_D^{(v)} & \text{for } (z_1, z_2) = (1, 0) \\ J_D^{(c)} & \text{for } (z_1, z_2) = (1, 1) \end{cases} \end{aligned} \quad (6.21)$$

It is straightforward to generalize (6.21) to the cases when the hypercolor group G is non-simple.

Thus, making the choices (6.21), we find that the total monodromy of \tilde{J} with respect to an arbitrary representation (R, r, \tilde{r}, Y) is precisely the left side of the charge-quantization condition (6.18). Since we are requiring that this is zero modulo 1, the locality condition on the current \tilde{J} is then automatically satisfied. Although not explicitly indicated in (6.20), the current \tilde{J} also contains an anti-holomorphic part, and we must choose this in such a way that it too is a local simple current. This is easily achieved if we take this current to be the *identity* primary field in the anti-analytic sector. As this has zero conformal dimension, the condition for modular invariance under $T : \tau \rightarrow \tau + 1$ becomes the condition that the conformal dimension of \tilde{J} in the analytic sector must itself also be an integer. Thus we discover that for a consistent theory, we must have

$$h(\tilde{J}) = h(J_G) + \frac{k_3}{3} + \frac{k_2}{4} + \frac{k_Y}{4} \in \mathbb{Z} \quad (6.22)$$

where $h(J_G)$ represents the conformal dimension of the relevant current J_G in (6.21). This conformal dimension can be generally calculated in terms of the conformal dimensions of the elementary simple currents given in (6.6), (6.9), (6.11), and (6.13). Hence, writing our arbitrary hypercolor gauge group in the general form

$$G \equiv \left[\prod_i SU(p_i)_{k_i} \right] \otimes \left[\prod_j SO(2q_j + 1)_{k_j} \right] \otimes \left[\prod_\ell Sp(2r_\ell)_{k_\ell} \right] \otimes \left[\prod_m SO(2s_m)_{k_m} \right], \quad (6.23)$$

we immediately obtain our final general condition on the Kač-Moody levels:

$$\sum_i \frac{zk_i(p_i - z)}{2p_i} + \sum_j \frac{k_j}{2} + \sum_\ell \frac{k_\ell q_\ell}{4} + \sum_m h_m + \frac{k_3}{3} + \frac{k_2}{4} + \frac{k_Y}{4} \equiv 0 \pmod{1}. \quad (6.24)$$

Note that in this equation, the quantity h_m should be taken to be $k_m s_m / 8$ if the quantization condition requires $J_D^{(s)}$ or $J_D^{(c)}$ in \tilde{J} , while we should take $h_m = k_m / 2$ if the quantization condition requires $J_D^{(v)}$. Both cases should be considered in order to allow for all possibilities. Likewise, the integer z which appears in the first term of (6.24) is arbitrary within the range $1 \leq z \leq p_i - 1$, and so once again all possibilities must be included. The corresponding integers z and (z_1, z_2) for the SO and Sp groups have been omitted from (6.24), since their only non-trivial allowed value is 1. Also note that despite the previous distinctions between the $SO(4m)$ and $SO(4m + 2)$ gauge groups, all $SO(2n)$ groups now contribute identically to this final constraint.

Eq. (6.24), then, is the general relation between the Kač-Moody levels that we have been seeking. In particular, this equation must be satisfied in any modular-invariant string model if that model is to contain only those states which will bind into color singlets under the influence of an extra arbitrary hypercolor gauge group of the form (6.23). It is thus straightforward to examine the possible cases of different hypercolor groups in order to determine which values of (k_Y, k_2, k_3) are mutually consistent.

The first special case to consider, of course, is that for which there is *no* hypercolor group at all, so that the only states which appear in the spectrum of a given string model are integer-charged directly. In this case, the condition (6.24) reduces to the condition originally obtained by Schellekens [23]:

$$\frac{k_3}{3} + \frac{k_2}{4} + \frac{k_Y}{4} \equiv 0 \pmod{1}. \quad (6.25)$$

It is easy to see, then, that for $k_2 = k_3 = 1$, the minimum allowed value of k_Y is indeed $k_Y = 5/3$. Thus, for level-one string models with only integer-charged states, we have

$$k_Y \geq 5/3. \quad (6.26)$$

In general, though, string models *do* contain fractionally charged states. In fact, as shown in Ref. [23], it is impossible to have a level-one $SU(3) \times SU(2) \times U(1)_Y$ string

model with $k_Y = 5/3$ *without* having fractionally charged states in the corresponding spectrum, for any GSO projections that would remove all of the fractionally charged states in such cases would also simultaneously promote the $SU(3) \times SU(2) \times U(1)_Y$ gauge symmetry to level-one $SU(5)$. We therefore seek to determine the more general k_Y constraints that arise from (6.24) when arbitrary hypercolor binding groups are present. More specifically, motivated by the phenomenologically preferred levels (3.19), we seek to know for which choices of binding groups G such levels may be accommodated without giving rise to unconfined fractionally charged states in the string spectrum.

Our procedure is as follows. For each value of $k_2 = k_3 = 1$ or 2 , we examine each possible choice of binding group G . Although the formalism presented above is completely general, we restrict ourselves in this analysis to the case of simple groups only. For each simple group, we then examine each possible rank and level subject to the requirements that the resulting group actually *confine*, and that it (along with the MSSM gauge factors) not exceed the total central charge $c_{\text{left}} = 22$ for the internal worldsheet theory of a consistent heterotic string. It turns out that there is also a periodicity in the allowed solutions for k_Y as a function of the level k of the confining group; for example, in the case of the $SO(2n+1)$ confining groups, we see from (6.24) that the allowed solutions for k_Y depend only on whether $k_{SO(2n+1)}$ is even or odd. We can therefore restrict ourselves, without loss of generality, to the lowest levels k appropriate for each confining group. Finally, in the case of the $SU(n)$ groups, we examine each of the cases with $z = 1, \dots, n-1$; we likewise examine the cases with $J_D^{(s,c)}$ and $J_D^{(v)}$ separately in the case of the $SO(2n)$ groups.

Our results are as follows. For the simplest case with $k_2 = k_3 = 1$, we find that there are only sixteen different binding scenarios which permit solutions with $k_Y < 5/3$. These confining groups, currents, and corresponding solutions for k_Y are listed below:

G	J_G	k_Y	G	J_G	k_Y
$SU(4)_3$	$J_A^{(1,3)}$	7/6	$SU(13)_1$	$J_A^{(3,10)}$	41/39
$SU(6)_5$	$J_A^{(1,5)}$	4/3	$SU(17)_1$	$J_A^{(7,10)}$	73/51*
$SU(7)_3$	$J_A^{(2,5)}$	23/21	$SU(17)_1$	$J_A^{(8,9)}$	61/51
$SU(9)_1$	$J_A^{(4,5)}$	11/9	$SU(18)_1$	$J_A^{(7,11)}$	10/9
$SU(10)_1$	$J_A^{(3,7)}$	22/15*	$SU(19)_1$	$J_A^{(6,13)}$	83/57*
$SU(10)_2$	$J_A^{(3,7)}$	19/15	$SO(10)_5$	$J_D^{(s,c)}$	7/6
$SU(11)_1$	$J_A^{(3,8)}$	43/33	$SO(18)_1$	$J_D^{(s,c)}$	7/6
$SU(12)_1$	$J_A^{(3,9)}$	7/6	$SO(34)_1$	$J_D^{(s,c)}$	7/6

We have indicated with an asterisk those solutions for which k_Y lies in the phenomenologically preferred range $1.4 \leq k_Y \leq 1.5$. We see, then, that for $k_2 = k_3 = 1$, the only binding scenarios permitting the phenomenologically preferred values of k_Y

are:

$$k_2 = k_3 = 1, \quad 1.4 \leq k_Y \leq 1.5 : \quad G = SU(10)_1, \quad SU(17)_1, \quad SU(19)_1 . \quad (6.28)$$

Of these limited choices with such large ranks, $SU(10)_1$ will probably be the most feasible hypercolor group to realize in a consistent string model.

For $k_2 = k_3 = 2$, we find that more solutions with $k_Y < 5/3$ are possible. Below we list only those hypercolor groups that permit values of k_Y in the range $1.4 \leq k_Y \leq 1.5$:

G	J_G	k_Y	G	J_G	k_Y	
$SU(4)_3$	$J_A^{(1,3)}$	$17/12$	$SU(12)_1$	$J_A^{(3,9)}$	$17/12$	(6.29)
$SU(6)_5$	$J_A^{(1,5)}$	$3/2$	$SU(17)_1$	$J_A^{(8,9)}$	$73/51$	
$SU(9)_1$	$J_A^{(4,5)}$	$13/9$	$SO(18)_1$	$J_D^{(s,c)}$	$17/12$	
$SU(10)_2$	$J_A^{(3,7)}$	$22/15$	$SO(34)_1$	$J_D^{(s,c)}$	$17/12$	
$SU(11)_1$	$J_A^{(3,8)}$	$49/33$				

Thus, for $k_2 = k_3 = 2$, we see that relatively small hypercolor groups can now confine the fractionally charged states that arise for $1.4 \leq k_Y \leq 1.5$. In particular, $SU(4)_3$ and $SU(6)_5$ are the most likely candidates for realization in a consistent higher-level string model. It is straightforward to continue this analysis to higher levels (k_2, k_3) and to non-simple (tensor-product) hypercolor groups.

6.4 Constraints with Fractional Charges

Another interesting variant of the Schellekens condition is to suppose that the gauge groups that couple to states with $U(1)_{\text{e.m.}}$ charge are just the MSSM gauge group factors (*i.e.*, that $G = 1$), but to allow levels that lead to *fractionally* charged asymptotic states. One would then hope that when realized in a consistent string model, such states can become superheavy and not appear in the low-energy effective theory. If we allow arbitrary $1/p$ multiples of the electron charge, then the quantization condition analogous to (6.18) is

$$p \left(\frac{c_{su(3)}(r)}{3} + \frac{c_{su(2)}(\tilde{r})}{2} + Y \right) \equiv 0 \pmod{1} . \quad (6.30)$$

Thus, if we define the residues p_2 and p_3 via $p \equiv p_m \pmod{m}$ with $0 \leq p_m < m$, then the simple current combination

$$\tilde{J} = \left(J_{SU(3)}^{(p_3)}, J_{SU(2)}^{(p_2)}, \exp(ip\sqrt{k_Y/2}\phi) \right) \quad (6.31)$$

satisfies the necessary locality property. Here ϕ , as in (6.20), is a free boson realizing the $U(1)_Y$ gauge symmetry. Computing the conformal dimension of \tilde{J} then gives the following constraint on the Kač-Moody levels (k_Y, k_2, k_3) :

$$2p_3(3-p_3)k_3 + 3p_2(2-p_2)k_2 + 3p^2k_Y \equiv 0 \pmod{12} . \quad (6.32)$$

It is straightforward to tabulate the allowed values of k_Y as a function of charge fractions p . Clearly, as p increases, the constraint (6.32) becomes weaker, so that increasingly many solutions with k_Y in any given range can be found. Therefore, motivated by (3.19), we seek solutions with k_Y in the narrow range $1.4 \leq k_Y/k_2 \leq 1.5$ for relatively small values of p and for $k_2 = k_3$.

Our results are as follows. For $k_2 = k_3 = 1$, we find that there are no suitable fractional charges which will allow $1.4 \leq k_Y \leq 1.5$ until $p = 4$, which permits the solution $k_Y = 17/12 \approx 1.417$. The next solution then appears for $p = 6$, and has the (somewhat better) value $k_Y = 13/9 \approx 1.444$. Solutions then appear for $p = 7$, and so forth.

For $k_2 = k_3 = 2$, by contrast, we find solutions in the desired range starting at $p = 3$, which yields $k_Y/k_2 = 13/9$. There are then solutions for $p = 4$ and $p = 5$, with values $k_Y/k_2 = 17/12$ and $k_Y/k_2 = 107/75 \approx 1.427$ respectively. Somewhat higher values of k_Y/k_2 do not appear until $p = 6$, which permits both $k_Y/k_2 = 13/9$ and $k_Y/k_2 = 3/2 = 1.5$. There are then suitable solutions for $p = 7$, and so forth. Likewise, for $k_2 = k_3 = 3$, there are again no solutions in the desired range until $p = 4$.

Interestingly, we find that phenomenologically viable solutions with *half*-integrally charged states ($p = 2$) do not occur until level $k_2 = k_3 = 4$, where we have the isolated solution $k_Y/k_2 = 17/12$. Thus, models with only integrally or half-integrally charged states cannot have values of k_Y in the desired range unless their MSSM factors are realized at level $k_2 = k_3 \geq 4$. At such levels, the next solution then appears for $p = 3$, with $k_Y/k_2 = 13/9$.

7 Conclusions and Discussion

In this paper, we have examined the extent to which the appearance of higher-level Kač-Moody gauge symmetries and non-standard hypercharge normalizations can provide a resolution to the gauge-coupling unification problem in string theory. We analyzed the phenomenological constraints on the allowed regions of (k_Y, k_2, k_3) parameter space, and found strong correlations between the values of the ratios $r \equiv k_Y/k_2$ and $r' \equiv k_3/k_2$ and the *absolute sizes* of the required levels (k_Y, k_2, k_3) . These results are summarized in Figs. 2 and 4. We then examined the possible hypercharge embeddings that might lead to exotic hypercharge normalizations with $k_Y < 5/3$, and found that for a specific class of realistic string models, such values of k_Y are impossible. This class of string models includes most of the realistic string models that exist in the literature to date. We also considered hypercharge embeddings beyond this class, however, and found that several isolated embeddings with $k_Y < 5/3$ exist. These embeddings involve higher-twist sectors giving rise to chiral matter, and/or higher-level MSSM gauge group factors. Finally, we considered the constraints on (k_Y, k_2, k_3) that arise from the requirement that no fractionally charged states appear in the string spectrum. We found that $k_Y \geq 5/3$ in all string models *without*

fractionally charged states, and determined which sets of additional hypercolor groups are capable of confining any fractionally charged states that appear for other values of k_Y . In this way we showed that only a relatively small set of hypercolor binding scenarios are possible for the phenomenologically preferred values of (k_Y, k_2, k_3) .

Our results raise a number of interesting issues which we shall now briefly discuss. First, as we mentioned above, it is clear that from the results of our renormalization-group analysis presented in Sect. 3 that there is a strong correlation between the Kač-Moody ratios $r \equiv k_Y/k_2$ and the absolute sizes of the Kač-Moody levels needed. This correlation ultimately stems from the intrinsically string-theoretic constraint that relates the coupling at unification to the unification scale. Thus, the Kač-Moody levels that are required for different values of k_Y/k_2 vary markedly, and are extremely sensitive to both experimental uncertainties (such as that in $\sin^2 \theta_W(M_Z)$), and theoretical corrections (such as those arising from heavy string thresholds, light SUSY thresholds, and intermediate matter thresholds). While we have not included these latter corrections in our analysis, their inclusion would be necessary before more precise statements concerning the phenomenologically preferred values of (k_Y, k_2, k_3) can be made. For example, we see from the charge-quantization analysis of Sect. 6 that the appearance of non-standard values of k_Y/k_2 is closely correlated to the appearance of additional fractionally-charged states in the string spectrum. These non-MSSM states have the potential to alter the running of the gauge couplings significantly. Similarly, while the effects of the heavy string threshold corrections are typically small in realistic string models (see, *e.g.*, the general arguments presented in Ref. [6]), it is unclear how we can expect these corrections to scale with the Kač-Moody levels (k_2, k_3) . A naive analysis would suggest that the gauge-dependent parts of these corrections should actually scale as k_i^{-1} (due to the shrinking of the roots of the charge lattice which accompanies the appearance of higher-level gauge symmetries), but it is possible that the subtle GSO projections that produce these higher-level gauge symmetries can, through modular invariance, add new twisted sectors that invalidate such naive estimates. Likewise, if $(k_Y/k_2, k_3/k_2)$ do not take their usual MSSM values $(5/3, 1)$, then there will also be contributions from the gauge-independent parts of these threshold corrections; these gauge-independent pieces are less well understood, and have the potential to be quite large [35]. Consequently, the answers to all of these questions are highly model-dependent, and their effects cannot be included in the sort of general framework we presented in Sect. 3. The results of Sect. 3 should nevertheless provide a suitable foundation upon which a realistic higher-level string model might be constructed.

Another immediate question raised by our analysis concerns the values of k_Y that are realizable in consistent string models, and the relation between the hypercharge embedding approach we followed in Sects. 4 and 5, and the charge-quantization approach we followed in Sect. 6. In the hypercharge embedding approach, we were able to prove that one must have $k_Y \geq 5/3$ for certain classes of string models. By contrast, beyond this class, we found that there exist special hypercharge embed-

dings which satisfy a host of string consistency constraints and succeed in realizing hypercharge normalizations $k_Y < 5/3$. In the charge-quantization approach, we likewise found a similar situation: for string models *without* fractionally charged states, we found that $k_Y \geq 5/3$, but smaller values of k_Y are potentially realizable if one allows fractionally charged states to appear and be confined under extra non-MSSM hypercolor interactions. Despite this superficial similarity, however, it is interesting to note that these two analyses did not yield identical results. In particular, our analysis of possible hypercharge embeddings in Sects. 4 and 5 yielded potential values of k_Y/k_2 which were, in many cases, different than those found to be realizable via the hypercolor scenarios we examined in Sect. 6. It is therefore natural to wonder if this observation implies that the special embeddings we constructed in Sect. 5 are not realizable in models which are phenomenologically acceptable (*i.e.*, free of observable fractionally charged states).

Unfortunately, this is also a difficult question to answer, for there are various subtleties that have not been taken into account. First, it is clear that the charge-integrality constraints derived in Sect. 6 apply to *all* string states, whether massive or massless, but such constraints might be evaded if we require only that the *massless* states be integrally charged. Unfortunately, it does not seem straightforward to incorporate this additional provision into the analysis. Likewise, another possibility would be to impose the even weaker requirement that only those massless states which are *chiral* must be integrally charged; after all, vector-like states which are fractionally charged and massless at tree level can acquire potentially large masses at higher loops (*e.g.*, via the shift of the moduli which is generally required in order to break anomalous $U(1)$ gauge symmetries and restore spacetime supersymmetry [36]). Once again, however, such scenarios are highly model-dependent, and therefore cannot be readily incorporated into the sort of general analysis we have been conducting.

Finally, it is also important to note that although we have restricted our analysis of $k_Y < 5/3$ hypercharge embeddings in Sects. 4 and 5 to those “minimal” embeddings in which the non-abelian MSSM gauge factors occupy only the first few components of the charge lattice, there may also exist alternative “extended” embeddings which make use of the additional lattice components [37]. Such embeddings also have the potential to yield hypercharge normalizations $k_Y < 5/3$ together with the MSSM matter content. However, since such embeddings make use of the non-MSSM components of the charge lattice, their structure is significantly less constrained than that of the embeddings we have studied here. Such embeddings are therefore best analyzed on a model-by-model basis.

Thus, we conclude that there exist various possible avenues by which higher-level Kač-Moody gauge symmetries and non-standard hypercharge normalizations can be employed to reconcile string-scale unification with the low-energy couplings. Indeed, given our general constraints concerning the allowed values of (k_Y, k_2, k_3) , the existence of hypercharge embeddings with $k_Y/k_2 < 5/3$, and the implications of such hypercharge normalizations for the appearance of fractionally charged string states,

it may prove possible to systematically construct realistic string models which not only realize the phenomenologically preferred values of (k_Y, k_2, k_3) , but which also simultaneously avoid unconfined fractionally charged states. These avenues therefore deserve further exploration.

Acknowledgments

We are pleased to thank I. Antoniadis, E. Witten, and especially S. Chaudhuri and J. Lykken for discussions. This work was supported in part by the Department of Energy under Grants No. DE-FG-0290ER40542 and DE-FG-0586ER40272, by the National Science Foundation under Grant No. PHY94-07194, and by the W.M. Keck Foundation.

Appendix A

In this Appendix, we briefly review the standard anomaly-cancellation arguments which fix (uniquely, up to an overall scale) the hypercharge assignments for the MSSM representations.

In this paper, we are considering theories that contain the field content of the $N = 1$ MSSM in their massless spectra. This means that the spectrum of any such theory must include the following $SU(3) \times SU(2) \times U(1)_Y$ $N = 1$ superfield representations:

$$\begin{aligned}
Q &\equiv (3, 2)_{y_Q} \\
u_R &\equiv (\bar{3}, 1)_{y_u} \\
d_R &\equiv (\bar{3}, 1)_{y_d} \\
L &\equiv (1, 2)_{y_L} \\
e_R &\equiv (1, 1)_{y_e} \\
H^+ &\equiv (1, 2)_{y_+} \\
H^- &\equiv (1, 2)_{y_-} .
\end{aligned} \tag{A.1}$$

We have left the seven hypercharge assignments $\{y_Q, y_u, y_d, y_L, y_e, y_+, y_-\}$ arbitrary, but in general these are subject to a variety of constraints. First, there are the constraints which arise from cancellation of gravitational and gauge anomalies:

$$\begin{aligned}
\text{gauge/gravitational} &\implies \text{Tr } Y = 0 \\
\text{pure gauge} &\implies \begin{cases} \text{Tr } Y^3 = 0 \\ \text{Tr } Y [J^{(2)}]^2 = 0 \\ \text{Tr } Y [J^{(3)}]^2 = 0 \end{cases}
\end{aligned} \tag{A.2}$$

where $J^{(2)}$ and $J^{(3)}$ refer to $SU(2)$ and $SU(3)$ currents respectively. Similarly, there are the constraints that come from the required $U(1)_Y$ -invariance of our desired mass

terms

$$Qu_R H^+ , \quad Qd_R H^- , \quad Le_R H^- , \quad H^+ H^- , \quad (\text{A.3})$$

namely that the sum of the hypercharges for each mass term in (A.3) must vanish. It turns out that only six of these eight constraint equations are linearly independent, however. For example, the sum of the $Qu_R H^+$ and $Qd_R H^-$ constraints reproduces the $\text{Tr } Y[J^{(3)}]^2$ constraint when the $H^+ H^-$ constraint is taken into account. Taken together, therefore, these equations constrain the seven hypercharge variables only up to an overall scale, with the fixed ratios

$$y_Q : y_u : y_d : y_L : y_e : y_+ : y_- = 1 : -4 : 2 : -3 : 6 : 3 : -3 . \quad (\text{A.4})$$

Choosing a scale so that the singlet representation e_R has unit hypercharge then yields the hypercharge assignments

$$\{y_Q, y_u, y_d, y_L, y_e, y_+, y_-\} \equiv \{1/6, -2/3, 1/3, -1/2, 1, 1/2, -1/2\} . \quad (\text{A.5})$$

These are of course the standard hypercharge assignments, and our point here has been to demonstrate that except for an overall scale, there is no freedom to adjust these assignments. Furthermore, fixing the scale by assigning unit hypercharge to the singlet representation is convenient since this assignment is independent of any normalizations, conventions, or Kač-Moody levels for the $SU(2)$ or $SU(3)$ gauge group factors in the model. This choice for the hypercharge scale is also the one that leads to the usual normalization factor $k_Y = 5/3$ for an $SU(5)$ or $SO(10)$ embedding. Thus, any consistent and realistic string model containing the MSSM spectrum (A.1) and gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$ must have an identifiable $U(1)_Y$ factor whose eigenvalues acting on these representations reproduces (A.5).

Appendix B

In this Appendix we demonstrate that, as claimed in Sect. 3, the two-loop corrections $\Delta_Y^{(2\text{-loop})}$, $\Delta_2^{(2\text{-loop})}$, and $\Delta_3^{(2\text{-loop})}$ depend only logarithmically on the ratios of the Kač-Moody levels k_Y , k_2 , and k_3 .

We begin by considering the full two-loop Yukawa-less renormalization group equations for the MSSM couplings

$$\frac{dg_i}{dt} = -\frac{g_i}{16\pi^2} \left[b_i g_i^2 + \frac{1}{16\pi^2} \sum_j b_{ij} g_i^2 g_j^2 \right] , \quad (\text{B.1})$$

which can be rewritten in terms of $\alpha_i \equiv g_i^2/(4\pi)$ as

$$\frac{d}{dt} \left(\frac{4\pi}{\alpha_i} \right) = 2b_i + \frac{1}{2\pi} \sum_j b_{ij} \alpha_j . \quad (\text{B.2})$$

Here b_i and b_{ij} are respectively the one- and two-loop beta-function coefficients, and $t \equiv \ln(M_{\text{string}}/M)$ where M is the variable setting the energy scale. If we neglect the two-loop term in (B.2), it is straightforward to integrate this equation from $M = \mu$ to $M = M_{\text{string}}$ to obtain the usual one-loop RGE

$$\frac{4\pi}{\alpha_i(\mu)} = \frac{4\pi}{\alpha_i(M_{\text{string}})} + 2b_i \ln \frac{M_{\text{string}}}{\mu}. \quad (\text{B.3})$$

Thus the two-loop “correction term”, obtained by taking the difference between the results of the one- and two-loop evolutions over this range, is simply

$$\Delta_i^{(2\text{-loop})} \equiv -\frac{1}{2\pi} \sum_j b_{ij} \int_{\mu}^{M_{\text{string}}} dt \alpha_j(t). \quad (\text{B.4})$$

We can approximate the value of this integral by substituting the one-loop result for $\alpha_j(t)$ from (B.3) into the integrand. We then have

$$\Delta_i^{(2\text{-loop})} = -\frac{1}{2\pi} \sum_j b_{ij} \alpha_j(M_{\text{string}}) \int_{\mu}^{M_{\text{string}}} dt \left[1 + \frac{b_j \alpha_j(M_{\text{string}})}{2\pi} t \right]^{-1}, \quad (\text{B.5})$$

which can be integrated analytically to yield

$$\Delta_i^{(2\text{-loop})} = \sum_j b_{ij} b_j^{-1} \ln \left[1 + \frac{b_j \alpha_j(M_{\text{string}})}{2\pi} \ln \frac{M_{\text{string}}}{\mu} \right]. \quad (\text{B.6})$$

Thus we see that $\Delta_i^{(2\text{-loop})}$ depends only logarithmically on the values of the $\alpha_i(M_{\text{string}})$. Now, M_{string} is defined as that scale at which the unification takes place, with each coupling related to the string coupling at tree-level via

$$\alpha_i(M_{\text{string}}) = \frac{\alpha_{\text{string}}(M_{\text{string}})}{k_i}. \quad (\text{B.7})$$

Thus we conclude that the dependence of the $\Delta_i^{(2\text{-loop})}$ on the levels k_i is also logarithmic:

$$\Delta_i^{(2\text{-loop})} = \sum_j b_{ij} b_j^{-1} \ln \left[1 + \frac{b_j \alpha_{\text{string}}(M_{\text{string}})}{2\pi k_j} \ln \frac{M_{\text{string}}}{\mu} \right]. \quad (\text{B.8})$$

In fact, it is straightforward to see from (B.8) that this logarithmic dependence involves only the *ratios* of the Kač-Moody levels, for if we take $\mu = M_Z$ and solve (3.13) for $\alpha_{\text{string}}(M_{\text{string}})$ in terms of the fixed low-energy electromagnetic coupling $a \equiv \alpha_{\text{e.m.}}(M_Z)$, we find

$$\Delta_i^{(2\text{-loop})} = \sum_j b_{ij} b_j^{-1} \ln \left[1 + \frac{k_Y + k_2}{k_j} \left(\frac{2\pi}{a b_j \ln(M_{\text{string}}/M_Z)} - \frac{b_Y + b_2}{b_j} \right)^{-1} \right]. \quad (\text{B.9})$$

Thus, neglecting the level-dependence of the scale M_{string} (which appears only as a double-log effect), we see that the leading dependence of the corrections $\Delta_i^{(2\text{-loop})}$ on the levels k_i is only through the *logarithm of their ratios*. Moreover, explicitly evaluating (B.9) for the MSSM spectrum and beta-function coefficients yields results which are close to the exact results quoted in (3.6), thereby verifying the validity of the approximation made by substituting the one-loop result (B.3) into the integrand of (B.4).

References

- [1] For a review, see:
M.B. Green, J.H. Schwarz, and E. Witten, *Superstring Theory, Vols. 1 & 2* (Cambridge University Press, Cambridge, 1987).
- [2] P. Ginsparg, *Phys. Lett.* **B197** (1987) 139.
- [3] M. Dine and N. Seiberg, *Phys. Rev. Lett.* **55** (1985) 366;
V.S. Kaplunovsky, *Phys. Rev. Lett.* **55** (1985) 1036.
- [4] V.S. Kaplunovsky, *Nucl. Phys.* **B307** (1988) 145; Erratum: *ibid.* **B382** (1992) 436, hep-th/9205070.
- [5] K.R. Dienes and A.E. Faraggi, *Phys. Rev. Lett.* **75** (1995) 2646, hep-th/9505018.
- [6] K.R. Dienes and A.E. Faraggi, hep-th/9505046 (to appear in *Nucl. Phys.* **B**).
- [7] See, *e.g.*, the following papers and references therein:
L.J. Dixon, V.S. Kaplunovsky, and J. Louis, *Nucl. Phys.* **B355** (1991) 649;
V.S. Kaplunovsky and J. Louis, *Nucl. Phys.* **B444** (1995) 191, hep-th/9502077;
L. Ibáñez, D. Lust, and G. Ross, *Phys. Lett.* **B272** (1991) 251;
L. Ibáñez and D. Lust, *Nucl. Phys.* **B382** (1992) 305;
I. Antoniadis, K.S. Narain, and T.R. Taylor, *Phys. Lett.* **B267** (1991) 37;
I. Antoniadis, E. Gava, and K.S. Narain, *Phys. Lett.* **B283** (1992) 209, hep-th/9203071;
J.P. Derendinger, S. Ferrara, C. Kounnas, and F. Zwirner, *Phys. Lett.* **B271** (1991) 307; *Nucl. Phys.* **B372** (1992) 145;
G. Lopes Cardoso and B.A. Ovrut, *Nucl. Phys.* **B369** (1992) 351;
P. Mayr and S. Stieberger, *Nucl. Phys.* **B407** (1993) 725, hep-th/9303017; *Nucl. Phys.* **B412** (1994) 502, hep-th/9304055;
D. Bailin and A. Love, *Phys. Lett.* **B292** (1992) 315;
D. Bailin, A. Love, W.A. Sabra, and S. Thomas, *Phys. Lett.* **B320** (1994) 21, hep-th/9309133; *Mod. Phys. Lett.* **A10** (1995) 337, hep-th/9407049.
- [8] A. Font, L. Ibáñez, D. Lüst, and F. Quevedo, *Phys. Lett.* **B245** (1990) 401;
M. Cvetič *et al.*, *Nucl. Phys.* **B361** (1991) 194;
B. de Carlos, J.A. Casas, and C. Muñoz, *Nucl. Phys.* **B399** (1993) 623.
- [9] See, *e.g.*, the following papers and references therein:
I. Antoniadis, J. Ellis, R. Lacaze, and D.V. Nanopoulos, *Phys. Lett.* **B268** (1991) 188;
P. Mayr, H.P. Nilles, and S. Stieberger, *Phys. Lett.* **B317** (1993) 53, hep-th/9307171;
L. Dolan and J.T. Liu, *Nucl. Phys.* **B387** (1992) 86, hep-th/9205094;

- D.M. Pierce, *Phys. Rev.* **D50** (1994) 6469, hep-th/9508178;
M. Chemtob, hep-th/9506018.
- [10] P. Mayr and S. Stieberger, hep-th/9412196; hep-th/9504129.
- [11] H.P. Nilles and S. Stieberger, hep-th/9510009.
- [12] D.C. Lewellen, *Nucl. Phys.* **B337** (1990) 61;
J. Ellis, J.L. Lopez, and D.V. Nanopoulos, *Phys. Lett.* **B245** (1990) 375;
A. Font, L.E. Ibáñez, and F. Quevedo, *Nucl. Phys.* **B345** (1990) 389;
S. Chaudhuri, S.-W. Chung, G. Hockney, and J. Lykken, hep-ph/9501361;
G. Aldazabal, A. Font, L.E. Ibáñez, and A.M. Uranga, hep-th/9410206;
G. Cleaver, hep-th/9506006;
D. Finnell, hep-th/9508073.
- [13] I. Antoniadis, J. Ellis, J. Hagelin, and D.V. Nanopoulos, *Phys. Lett.* **B231** (1989) 65;
J. Lopez, D.V. Nanopoulos, and K. Yuan, *Nucl. Phys.* **B399** (1993) 654, hep-th/9203025.
- [14] S.M. Barr, *Phys. Lett.* **B112** (1982) 219; *Phys. Rev.* **D40** (1989) 2457;
J.P. Derendinger, J.E. Kim, and D.V. Nanopoulos, *Phys. Lett.* **B139** (1984) 170.
- [15] I. Antoniadis, G.K. Leontaris, and J. Rizos, *Phys. Lett.* **B245** (1990) 161.
- [16] J.C. Pati and A. Salam, *Phys. Rev.* **D8** (1973) 1240; *Phys. Rev. Lett.* **31** (1973) 661; *Phys. Rev.* **D10** (1974) 275.
- [17] A.E. Faraggi, *Phys. Lett.* **B339** (1994) 223, hep-ph/9408333.
- [18] I. Antoniadis, J. Ellis, S. Kelley, and D.V. Nanopoulos, *Phys. Lett.* **B272** (1991) 31;
A.E. Faraggi, *Phys. Lett.* **B302** (1993) 202, hep-ph/9301268.
- [19] M.K. Gaillard and R. Xiu, *Phys. Lett.* **B296**, 71 (1992);
S.P. Martin and P. Ramond, *Phys. Rev.* **D51**, 6515 (1995).
- [20] C. Bachas, C. Fabre, and T. Yanagida, hep-th/9510094.
- [21] J.A. Casas and C. Munoz, *Phys. Lett.* **B214** (1988) 543;
L. Ibáñez, *Phys. Lett.* **B318** (1993) 73, hep-ph/9408333.
- [22] See, *e.g.*, H. Kawabe, T. Kobayashi, and N. Ohtsubo, *Nucl. Phys.* **B434** (1995) 210, and references therein.
- [23] A.N. Schellekens, *Phys. Lett.* **B237** (1990) 363.

- [24] For a pedagogical introduction, see, *e.g.*:
P. Ginsparg, *Applied Conformal Field Theory*, published in *Fields, Strings, and Critical Phenomena: Proceedings of Les Houches, Session XLIX, 1988* (Elsevier, 1989).
- [25] See, *e.g.*:
Particle Data Group, L. Montanet *et al*, *Phys. Rev.* **D50** (1994) 1173;
P. Langacker, hep-ph/9412361, and references therein;
N. Polonsky, hep-ph/9411378, and references therein.
- [26] A.E. Faraggi, *Phys. Lett.* **B278** (1992) 131; *Phys. Lett.* **B274** (1992) 47; *Nucl. Phys.* **B387** (1992) 239, hep-th/9208024.
- [27] D. Bailin and A. Love, *Phys. Lett.* **B203** (1988) 237.
- [28] P.F. Smith, *Ann. Rev. Nucl. Part. Sci.* **39**, (1989) 73.
- [29] J. Ellis, J. Lopez, and D. Nanopoulos, *Phys. Lett.* **B245** (1990) 375.
- [30] I. Antoniadis and K. Benakli, *Phys. Lett.* **B295** (1992) 219.
- [31] A.N. Schellekens and S. Yankielowicz, *Nucl. Phys.* **B327** (1989) 673.
- [32] K. Intriligator, *Nucl. Phys.* **B332** (1990) 541.
- [33] D. Versteegen, *Nucl. Phys.* **B346** (1990) 349.
- [34] J. Fuchs, *Affine Lie Algebras and Quantum Groups* (Cambridge University Press, Cambridge, England, 1992).
- [35] I. Antoniadis, E. Gava, and K.S. Narain, *Nucl. Phys.* **B383** (1992) 93, hep-th/9204030;
B. de Wit, V.S. Kaplunovsky, J. Louis, and D. Lüst, hep-th/9504006;
I. Antoniadis *et al*, hep-th/9504034.
- [36] M. Dine, N. Seiberg, and E. Witten, *Nucl. Phys.* **B289** (1987) 589;
J.J. Atick, L.J. Dixon, and A. Sen, *Nucl. Phys.* **B292** (1987) 109;
S. Cecotti, S. Ferrara, and M. Villasante, *Int. J. Mod. Phys.* **A2** (1987) 1839.
- [37] S. Chaudhuri, G. Hockney, and J. Lykken, hep-th/9510241.